Universidade Federal do Rio de Janeiro
Centro de Ciências Matemáticas e da Natureza
Instituto de Matemática

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Bernardo Melo de Carvalho

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Tese de Doutorado apresentada ao Programa de Pós-Graduação em Matemática do Instituto de Matemática da Universidade Federal do Rio de Janeiro, como parte dos requisitos necessários à obtenção do Título de Doutor em Matemática.

Orientador: Prof. Dr. Alexander Eduardo Arbieto Mendoza

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The two-sided limit shadowing property

Bernardo Melo de Carvalho

Advisor: Alexander Eduardo Arbieto Mendoza

In this thesis we study in detail the two-sided limit shadowing property in the theory of dynamical systems. We discuss some of its consequences and its role in the hyperbolic theory. This property is among the strongest known pseudo-orbit tracing properties. It implies shadowing, average shadowing, asymptotic average shadowing and even the specification property. It is present in the transitive Anosov scenario and characterizes two important conjectures in the hyperbolic theory. We present a new approach to these conjectures. In the appendix we exhibit a joint-work with A. Arbieto, W. Cordeiro and D. Obata named "On bi-Lyapunov stable homoclinic classes".

A propriedade sombreamento no limite para os dois lados

Bernardo Melo de Carvalho

Orientador: Alexander Eduardo Arbieto Mendoza

Nessa tese estudamos em detalhe a propriedade sombreamento no limite para os dois lados na teoria de sistemas dinâmicos. Discutimos algumas de suas consequências e o seu papel na teoria hiperbólica. Ela está entre as noções mais fortes dentre as propriedades de sombreamento, implicando sombreamento, sombreamento em média, sombreamento em média assintótico e até a propriedade de especificação. Ela está presente nos difeomorfismos de Anosov transitivos e caracteriza duas conjecturas importantes na teoria hiperbólica. Introduzimos novas ideias de como resolver essas conjecturas. No apêndice exibimos um trabalho em conjunto com A. Arbieto, W. Cordeiro e D. Obata chamado “On bi-Lyapunov stable homoclinic classes”.

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Contents

Chapter 1

Introduction

The theory of shadowing has been intensively developed in recent years. It is of extreme importance in the qualitative study of dynamical systems. Apart of the now standard shadowing property various variants of this concept were proposed. These new properties arise from modifications of the notion of pseudo-orbit together with different definitions of shadowing points. The main property we consider in this text is called *two-sided limit shadowing.*

Definition 1. *We say that a bi-lateral sequence* $(x_k)_{k\in\mathbb{Z}}$ *of points from a metric space* (X,d) *is a* two-sided limit pseudo-orbit *for a homeomorphism* $f\colon X\to X$ *if it satisfies*

$$d(f(x_k),x_{k+1})\to 0, \quad |k|\to\infty.$$

A sequence $(x_k)_{k\in\mathbb{Z}}\subset X$ *is* two-sided limit shadowed *if there exists* $y\in X$ *satisfying*

$$d(f^k(y),x_k)\to 0, \quad |k|\to\infty.$$

In this situation we also say that y two-sided limit shadows $(x_k)_{k\in\mathbb{Z}}$ *with respect to* f*. Finally, we say that* f *has the* two-sided limit shadowing property *if every two-sided limit pseudo-orbit is two-sided limit shadowed.*

following: if we apply a numerical method that approximate f with 'improving accuracy' so that one step errors tend to zero as time goes to infinity then the numerically obtained orbits tend to real ones.

In this thesis we analyze some consequences of the two-sided limit shadowing property and relate it to well known open problems in hyperbolic theory. The main theorems here belong to a series of three papers about this property ([15], [16] and [17]). In this text, however, we do not follow the chronological order of these works, neither display the papers as they appear in the journals. We make some adjustments to the original works, we change the structure of the papers and add some interesting properties and questions that clarify the results. Although the proofs are essentially the same.

We begin with a natural discussion about the difference of the two-sided limit shadowing property and the one-sided version of limit shadowing, that we will call the *limit shadowing property.* It was introduced by T. Eirola, O. Nevanlinna and S. Pilyugin in [19] and is also studied in [48] and [49].

Definition 2. *By* $\mathbb{N}$ *we denote the set of positive integers and* $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. *We say that* $(x_k)_{k\in\mathbb{N}_0}$ *is a* limit pseudo-orbit (for f) *if it satisfies*

$$d(f(x_k), x_{k+1}) \to 0, \quad k \to \infty.$$

A sequence $\{x_k\}_{k\in\mathbb{N}_0}$ *is* limit-shadowed *if there exists* $y \in X$ *such that*

$$d(f^k(y), x_k) \to 0, \quad k \to \infty.$$

We say that f *has the* limit shadowing property *if every limit pseudo-orbit is limit-shadowed.*

One of our goals is to understand the differences between these two properties: the limit shadowing and the two-sided limit shadowing. We prove the latter is among the strongest known notions of pseudo-orbit tracing properties.

Theorem 1 ([17])**.** *If a homeomorphism on a compact metric space has the two-sided limit shadowing property then it has shadowing, average shadowing, asymptotic average shadowing, it is topologically mixing and has the specification property.*

Precise definitions are given in Chapter 2. The relation between the two-sided limit shadowing property and hyperbolicity is one of our main interests and will be explored in detail along the text. We rewrite two important conjectures on Anosov diffeomorphisms in terms of the two-sided limit shadowing property.

According to conjectures of S. Smale [56] the following three classes of diffeomorphisms are expected to be the same:

- Anosov diffeomorphisms,
- transitive Anosov diffeomorphisms,[1]
- product Anosov diffeomorphisms.[2]

We write these classes in terms of the two-sided limit shadowing property.

[1] We say that a homeomorphism f on a metric space X is *transitive* if given any pair (U, V) of nonempty open subsets of X there exists $N \in \mathbb{N}$ such that $f^N(U) \cap V \neq \emptyset$.

[2] An Anosov diffeomorphism is a *product Anosov diffeomorphism* when each stable leaf and each unstable leaf on the universal covering intersect in a single point (see Chapter 4 for precise definitions).

Theorem 2 ([15],[16]). *An Anosov diffeomorphism on a compact and connected manifold is transitive if and only if it has the two-sided limit shadowing property. Furthermore, an Anosov diffeomorphism is a product Anosov diffeomorphism if and only if any lift to the universal covering has the unique two-sided limit shadowing property*[3].

So the following three classes of diffeomorphisms are expected to be the same:

- Anosov diffeomorphisms,
- Anosov diffeomorphisms with the two-sided limit shadowing property,
- Anosov diffeomorphisms with the unique two-sided limit shadowing property on the universal covering.

We note that Smale's conjecture is proved for *codimension one* Anosov diffeomorphisms[4] (see [21] and [46]). Hence the following is a corollary of Theorem 2.

Corollary 1. *Codimension one Anosov diffeomorphisms have the two-sided limit shadowing property and any lift to the universal covering has the unique two-sided limit shadowing property.*

Theorem 2 says to us that the shadowing theory might play an important role in these conjectures. So we use some ideas contained in the proof of the Shadowing Lemma of [49] and introduce some new techniques: for any two-sided limit pseudo-orbit $(x_k)_{k\in\mathbb{Z}}$ in the universal covering we discuss two maps F and G from a suitable Banach space to itself such that fixed points of these maps are related with points that two-sided limit shadows $(x_k)_{k\in\mathbb{Z}}$. Precisely, the following is proved:

Theorem 3 ([16]). *There exists a bijection between the set of points that two-sided limit shadows* $(x_k)_{k\in\mathbb{Z}}$ *and the set of fixed points of the maps* F *and* G.

These maps, and also the Banach space, are defined at the end of Chapter 4. In Chapter 5 we turn our attention to the two-sided limit shadowing property outside the Anosov scenario. Our main result in this setting says that the two-sided limit shadowing property is very uncommon in the set of all diffeomorphisms:

Theorem 4 ([15]). *There exists a residual subset*[5] *of the set of all diffeomorphisms on a compact manifold such that every diffeomorphism in this residual that is not a transitive Anosov diffeomorphism does not admit the two-sided limit shadowing property.*

[3] We say that a homeomorphism has the *unique two-sided limit shadowing property* when every two-sided limit pseudo-orbit is two-sided limit shadowed by a *single point*.

[4] We say that an Anosov diffeomorphism is a *codimension one Anosov diffeomorphism* if the dimension of either the stable direction or the unstable direction equals to one.

[5] We say that a subset of a topological space X is a *residual subset* if it contains a countable intersection of open and dense subsets of X.

In the Appendix we exhibit a joint work with A. Arbieto, W. Cordeiro and D. Obata named *Bi-Lyapunov stable homoclinic classes.* We discuss the hyperbolicity of homoclinic classes satisfying some of the shadowing properties discussed in the text. In this case we display the paper exactly as it is published in [9].

Chapter 2

Consequences of two-sided limit shadowing

In this Chapter we obtain many consequences of the two-sided limit shadowing property and prove Theorem 1.

2.1 Negative limit shadowing

We begin with a discussion about the limit shadowing property for the inverse map. From now on (X, d) denotes a metric space (not necessarily compact) and $f\colon X \to X$ denotes a homeomorphism. The limit shadowing property for f^{-1} means the following: for every sequence $(x_k)_{k\in\mathbb{N}_0} \subset X$ satisfying

$$d(f^{-1}(x_k), x_{k+1}) \to 0, \quad k \to \infty$$

there exists $z \in X$ satisfying

$$d(f^{-k}(z), x_k) \to 0, \quad k \to \infty.$$

During the text we will need the following property: for every sequence $(x_k)_{k\in-\mathbb{N}_0} \subset X$ ($-\mathbb{N}_0$ denotes the set of non-positive integers) satisfying

$$d(f^{-1}(x_k), x_{k-1}) \to 0, \quad k \to -\infty$$

there exists $z \in X$ satisfying

$$d(f^{k}(z), x_k) \to 0, \quad k \to -\infty.$$

Definition 3. *This property will be called* negative limit shadowing *and the sequence* $(x_k)_{k\in-\mathbb{N}_0}$ *will be called a* negative limit pseudo-orbit *for* f. *We will say that* z limit shadows $(x_k)_{k\in-\mathbb{N}_0}$ in the past *and that* $(x_k)_{k\in-\mathbb{N}_0}$ *is* limit shadowed in the past *by* z.

Proposition 1. *A homeomorphism f on a metric space X has the negative limit shadowing property if and only if f^{-1} has the limit shadowing property.*

Proof. Suppose first that f has the negative limit shadowing property and consider $(x_k)_{k\in\mathbb{N}_0}$ a limit pseudo-orbit for f^{-1}. For each $k \in -\mathbb{N}_0$ let $y_k = x_{-k}$. The sequence $(y_k)_{k\in-\mathbb{N}_0}$ is a negative limit pseudo-orbit for f since

$$d(f^{-1}(y_k), y_{k-1}) = d(f^{-1}(x_{-k}), x_{-k+1}) \to 0, \quad k \to -\infty.$$

Then there exists $z \in X$ satisfying

$$d(f^k(z), y_k) \to 0, \quad k \to -\infty$$

which implies

$$d(f^{-k}(z), x_k) \to 0, \quad k \to \infty.$$

It follows that $(x_k)_{k\in\mathbb{N}_0}$ is limit shadowed for f^{-1} and that f^{-1} has the limit shadowing property.

Now suppose that f^{-1} has the limit shadowing property and consider $(x_k)_{k\in-\mathbb{N}_0}$ a negative limit pseudo-orbit for f. For each $k \in \mathbb{N}_0$ let $y_k = x_{-k}$. The sequence $(y_k)_{k\in\mathbb{N}_0}$ is a limit pseudo-orbit for f^{-1} since

$$d(f^{-1}(y_k), y_{k+1}) = d(f^{-1}(x_{-k}), x_{-k-1}) \to 0, \quad k \to \infty.$$

Then there exists $z \in X$ satisfying

$$d(f^{-k}(z), y_k) \to 0, \quad k \to \infty$$

which implies

$$d(f^k(z), x_k) \to 0, \quad k \to -\infty.$$

It follows that $(x_k)_{k\in-\mathbb{N}_0}$ is limit shadowed in the past and that f has the negative limit shadowing property. □

We will use this fact during the text without referring to it. When we say that f^{-1} has the limit shadowing property we will usually use the fact that f has the negative limit shadowing property.

Question 1. *Is the limit shadowing property equivalent to the negative limit shadowing property?*

2.2 Simple two-sided limit shadowing with a gap

Toward proving Theorem 1 we try to exhibit all arguments in the most general way, so we find it convenient to generalize the two-sided limit shadowing property and introduce two different notions: the *simple two-sided limit shadowing property* and the *two-sided limit shadowing property with a gap*.

Definition 4. *For $x, y \in X$ we consider the sequence $(x_k)_{k\in\mathbb{Z}}$ defined by*

$$x_k = \begin{cases} f^k(y), & k \geq 0; \\ f^k(x), & k < 0. \end{cases}$$

This sequence consists of the past orbit *of x and the* future orbit *of y. Sequences of this type will be called* simple two-sided limit pseudo-orbits*. We say that f has the* simple two-sided limit shadowing property *when every simple two-sided limit pseudo-orbit is two-sided limit shadowed.*

It is obvious that the two-sided limit shadowing property implies the simple two-sided limit shadowing property. The converse is false as we will see in Corollary 6. Now we prove the following equivalence:

Lemma 1. *A homeomorphism on a metric space has the two-sided limit shadowing property if and only if it has the limit shadowing property, the negative limit shadowing property and the simple two-sided limit shadowing property.*

Proof. It is obvious that the two-sided limit shadowing property implies the limit shadowing property, the negative limit shadowing property and the simple two-sided limit shadowing property. It suffices to prove the converse statement. Let $(x_k)_{k\in\mathbb{Z}}$ be a two-sided limit pseudo-orbit for f. The limit shadowing property and the negative limit shadowing property assure the existence of points $z_1, z_2 \in X$ satisfying

$$d(f^k(z_1), x_k) \to 0, \quad k \to -\infty \quad \text{and} \quad d(f^k(z_2), x_k) \to 0, \quad k \to \infty.$$

Thus the sequence

$$y_k = \begin{cases} f^k(z_2), & k \geq 0 \\ f^k(z_1), & k < 0 \end{cases}$$

is a simple two-sided limit pseudo-orbit. The simple two-sided limit shadowing property assures the existence of a point $z \in X$ that two-sided limit shadows $(y_k)_{k\in\mathbb{Z}}$. This point also two-sided limit shadows $(x_k)_{k\in\mathbb{Z}}$. □

Corollary 2. *If a homeomorphism has the limit shadowing and the negative limit shadowing properties then it has the two-sided limit shadowing property if and only if it has the simple two-sided limit shadowing property.*

Now we define the two-sided limit shadowing property with a gap.

Definition 5. *We say that a sequence $(x_k)_{k\in\mathbb{Z}}$ is* two-sided limit shadowed with gap $K \in \mathbb{Z}$ for f *if there exists a point $y \in X$ satisfying*

$$d(f^k(y), x_k) \to 0, \quad k \to -\infty,$$
$$d(f^{K+k}(y), x_k) \to 0, \quad k \to \infty.$$

For $N \in \mathbb{N}_0$ we say that f has the two-sided limit shadowing property with gap N *if every two-sided limit pseudo-orbit of f is two sided limit shadowed with gap $K \in \mathbb{Z}$, where $|K| \leq N$. We also say that f has the* two-sided limit shadowing property with a gap *if such an $N \in \mathbb{N}$ exists.*

It is obvious that the two-sided limit shadowing property with gap 0 is equivalent to the two-sided limit shadowing property. It is also obvious that the two-sided limit shadowing property implies two-sided limit shadowing with a gap. The converse is false as the next example shows.

Example. *Consider the set $X = \{a, b\}$ endowed with the discrete metric*

$$d(x,y) = \begin{cases} 1, & x \neq y; \\ 0, & x = y. \end{cases}$$

Define $f\colon X \to X$ by $f(a) = b$ and $f(b) = a$. This map is a homeomorphism of X which does not have the simple two-sided limit shadowing property. Indeed, the sequence

$$x_k = \begin{cases} f^k(b), & k \geq 0; \\ f^k(a), & k < 0 \end{cases}$$

is a simple two-sided limit pseudo-orbit that cannot be two-sided limit shadowed. We note that f has the two-sided limit shadowing property with gap 1. By properties of the discrete metric we have that for each limit pseudo-orbit $(x_k)_{k\in\mathbb{N}} \subset X$ there exists $N \in \mathbb{N}$ such that $(x_k)_{k\geq N}$ is the future orbit of x_N. So for each two-sided limit pseudo-orbit $(x_k)_{k\in\mathbb{Z}}$ there exist $N_1, N_2 \in \mathbb{N}$ such that $(x_k)_{k\leq -N_2}$ is the past orbit of x_{-N_2} and $(x_k)_{k\geq N_1}$ is the future orbit of x_{N_1}. We can choose N_1 and N_2 such that $x_{N_1} = x_{-N_2} = a$. It follows that if $N_1 + N_2 = 0 \mod 2$ then either a or b two-sided limit shadows $(x_k)_{k\in\mathbb{Z}}$. Otherwise, (if $N_1 + N_2 = 1 \mod 2$) neither a, nor b, can two-sided limit shadows $(x_k)_{k\in\mathbb{Z}}$, but it is easy to check that one of them two-sided limit shadows $(x_k)_{k\in\mathbb{Z}}$ with gap 1.

Remark 1. *We note that this example has the limit shadowing property and the negative limit shadowing property but do not have the two-sided limit shadowing property, nor the simple two-sided limit shadowing property.*

Definition 6. *Let $I \subset \mathbb{Z}$ be a nonempty set of consecutive integers. We say that a sequence $(x_k)_{k\in I} \subset X$ is a* δ-pseudo-orbit (for f) *if it satisfies*

$$d(f(x_k), x_{k+1}) < \delta, \quad \text{for all } k \text{ such that } k, k+1 \in I.$$

We call a pseudo-orbit finite (positive, negative, two-sided) *if I is finite ($I = \mathbb{N}_0$, $I = -\mathbb{N}_0$, $I = \mathbb{Z}$, respectively). A sequence $(x_k)_{k\in I} \subset X$ is* ε-shadowed (with respect to f) *if there exists $y \in X$ satisfying*

$$d(f^k(y), x_k) < \varepsilon, \quad \text{for all } k \in I.$$

From now on we omit references to f when it is clear from the context which f we have in mind. We say that f has the shadowing property *if for every $\varepsilon > 0$ there exists $\delta > 0$ such that every two-sided δ-pseudo-orbit is ε-shadowed.*

When X is a compact metric space and f is a continuous map then one may replace *two-sided* by *one-sided* or by *finite* in the definition of the shadowing property without altering the notion (we leave the details to the reader). Excellent references for the shadowing property are Pilyugin's book [49] and Aoki and Hiraide's monograph [7]. The main result of this section is the following.

Theorem 5 ([17]). *If a homeomorphism on a compact metric space has the two-sided limit shadowing property with a gap then it is transitive and has the shadowing property.*

Before proving this theorem we state one definition and some results.

Definition 7. *For $N \in \mathbb{N}_0$ we say that a homeomorphism has the* simple two-sided limit shadowing property with gap N *when every simple two-sided limit pseudo-orbit is two-sided limit shadowed with gap $K \in \mathbb{Z}$ where $|K| \leq N$.*

It is obvious that the two-sided limit shadowing property with a gap implies the simple two-sided limit shadowing property with a gap. Now we prove that the simple two-sided limit shadowing property with a gap implies chain-transitivity. Recall that f is *chain-transitive* if for every $\delta > 0$ and every pair of points (x, y) in $X \times X$ there is a finite δ-pseudo-orbit $(z_k)_{k=0}^n$ such that $z_0 = x$ and $z_n = y$.

Lemma 2. *If a homeomorphism on a compact metric space has the simple two-sided limit shadowing property with a gap then it is chain-transitive.*

Proof. For each $x, y \in X$ and any $\delta > 0$ we want to find a δ-pseudo-orbit starting at x and ending at y. Consider the omega-limit set of x and the alpha-limit set of y, denoted by $\omega(x)$ and $\alpha(y)$, respectively. Since X is compact we can choose $z_1 \in \omega(x)$ and $z_2 \in \alpha(y)$. The hypothesis assures the existence of an integer K and a point z which two-sided limit shadows with gap K the sequence

$$x_k = \begin{cases} f^k(z_2), & k \geq 0; \\ f^k(z_1), & k < 0. \end{cases}$$

Thus there exists $M > K$ such that

$$d(f^{-M}(z), f^{-M}(z_1)) < \frac{\delta}{2} \quad \text{and} \quad d(f^M(z), f^{M-K}(z_2)) < \frac{\delta}{2}.$$

Since $\omega(x)$ and $\alpha(y)$ are compact and invariant subsets of X it follows that $f^{-M}(z_1) \in \omega(x)$ and $f^{M-K}(z_2) \in \alpha(y)$. Thus we can choose positive integers M_1 and M_2 such that

$$d(f^{M_1}(x), f^{-M}(z_1)) < \frac{\delta}{2} \quad \text{and} \quad d(f^{-M_2}(y), f^{M-K}(z_2)) < \frac{\delta}{2}.$$

Therefore

$$d(f^{M_1}(x), f^{-M}(z)) < \delta \quad \text{and} \quad d(f^{-M_2}(y), f^M(z)) < \delta.$$

Now consider the following sequence

$$y_n = \begin{cases} f^n(x), & \text{for } n = 0, \dots, M_1 - 1, \\ f^{n-M_1}(f^{-M}(z)), & \text{for } n = M_1, \dots, M_1 + 2M - 1, \\ f^{n-(M_1+2M)}(f^{-M_2}(y)), & \text{for } n = M_1 + 2M, \dots, M_1 + 2M + M_2. \end{cases}$$

Elements of this sequence are ordered as follows:

$$x, f(x), \dots, f^{M_1-1}(x), f^{-M}(z), f^{-M+1}(z), \dots, f^{M-1}(z), f^{-M_2}(y), \dots, y.$$

It is clear that $(y_n)_{n=0}^{M_1+2M+M_2}$ is a finite δ-pseudo-orbit connecting x to y. Since this can be done for any $\delta > 0$ we conclude the proof. □

In [35] it is proved that shadowing follows from chain-transitivity and the limit shadowing property. Following [35] we present a simple proof of this fact for completeness.

Theorem 6 ([35, Theorem 7.3]). *If a continuous map on a compact metric space is chain-transitive and has the limit shadowing property then it has the shadowing property.*

Proof. Aiming for a contradiction, suppose that f does not have the shadowing property. Hence there is $\varepsilon > 0$ such that for any $n > 0$ there is a finite $\frac{1}{n}$-pseudo-orbit α_n which cannot be ε-shadowed by any point in X. Using chain transitivity, for every n there exists a $\frac{1}{n}$-pseudo-orbit β_n such that the concatenated sequence $\alpha_n\beta_n\alpha_{n+1}$ forms a finite $\frac{1}{n}$-pseudo-orbit. Then the infinite concatenation

$$\alpha_1\beta_1\alpha_2\beta_2\alpha_3\beta_3\dots$$

is a limit pseudo-orbit denoted by $(x_k)_{k=0}^{\infty}$. By the limit shadowing property, it is limit shadowed by some point $z \in X$. Hence, starting at some index $N > 0$, the point $f^N(z)$ also ε-shadows the limit pseudo orbit $(x_k)_{k=N}^{\infty}$. But this means that there is a finite pseudo-orbit α_n which is ε-shadowed by some point of the form $f^k(z)$, which is a contradiction. □

The proof of the next two lemmas are easy exercises, hence we omit them.

Lemma 3. *If a continuous map on a metric space is chain-transitive and has the shadowing property then it is transitive.*

Lemma 4. *If a homeomorphism f has the two-sided limit shadowing property with a gap then f and f^{-1} have the limit shadowing property.*

Proof of Theorem 5: Assuming that f has the two-sided limit shadowing property with a gap then Lemma 4 assures f has the limit shadowing property and Lemma 2 assures f is chain-transitive. Thus Theorem 6 assures f has the shadowing property and Lemma 3 assures f is transitive.

2.3 Two-sided limit shadowing and specification

Before proving Theorem 1 we recall some definitions and results.

Definition 8. *A map f is* topologically mixing *if for every pair (U, V) of nonempty open subsets of X there exists $N \in \mathbb{N}$ such that $f^k(U) \cap V \neq \emptyset$ for all $k \geq N$.*

Definition 9. *Let $\tau = \{I_1, \dots, I_m\}$ be a finite collection of disjoint finite subsets of consecutive integers, $I_j = [a_j, b_j] \cap \mathbb{Z}$ for some $a_j, b_j \in \mathbb{Z}$, with*

$$a_1 \le b_1 < a_2 \le b_2 < \ldots < a_m \le b_m.$$

Let a map $P\colon \bigcup_{j=1}^{m} I_j \to X$ be such that for each $I \in \tau$ and $t_1, t_2 \in I$ we have

$$f^{t_2 - t_1}(P(t_1)) = P(t_2).$$

We call a pair (τ, P) a specification. *We say that the specification $S = (\tau, P)$ is L-*spaced *if $a_{j+1} \ge b_j + L$ for all $j \in \{1, \dots, m-1\}$. Moreover, S is ε-*shadowed *by $y \in X$ for f if*

$$d(f^k(y), P(k)) < \varepsilon \quad \textit{for every } k \in \bigcup_{j=1}^{m} I_j.$$

We say that a homeomorphism $f\colon X \to X$ has the specification property *if for every $\varepsilon > 0$ there exists $L \in \mathbb{N}$ such that every L-spaced specification is ε-shadowed. We say that f has the* periodic specification property *if for every $\varepsilon > 0$ there exists $L \in \mathbb{N}$ such that every L-spaced specification is ε-shadowed by a periodic point y such that $f^{b_m + L}(y) = y$.*

It is well known that mixing is a necessary condition for specification. We prove this fact for completeness.

Lemma 5. *If a homeomorphism on a compact metric space has the specification property, then it is topologically mixing.*

Proof. For each pair (U, V) of non-empty open subsets of X choose points $x \in U$ and $y \in V$ and consider $\varepsilon > 0$ such that the ε-neighborhood of x is contained in U and the ε-neighborhood of y is contained in V. The specification property assures the existence of a number $L \in \mathbb{N}$ such that every L-spaced specification is ε-shadowed. Thus for each $n \ge L$ consider the specification (τ_n, P_n) defined by $\tau_n = \{\{0\}, \{n\}\}$, $P_n(0) = x$ and $P_n(n) = y$. Then there exists $z_n \in X$ such that $d(x, z_n) < \varepsilon$ and $d(y, f^n(z_n)) < \varepsilon$. In particular $z_n \in U$ and $f^n(z_n) \in V$. Since this holds for all $n \ge L$ it follows that f is topologically mixing. □

The specification property can be obtained by mixing and shadowing as follows.

Lemma 6 ([18])**.** *If a homeomorphism on a compact metric space is topologically mixing and has the shadowing property then it has the specification property.*

Proof. Consider an arbitrary $\varepsilon > 0$ and choose $\delta > 0$ such that every δ-pseudo-orbit is ε-shadowed. Let $\mathcal{U} = \{U_1, \dots, U_m\}$ be a finite cover of X by open balls of radius δ. Since f is topologically mixing, for each pair $(U_i, U_j) \in \mathcal{U} \times \mathcal{U}$ there exists an integer $L_{i,j} > 0$ such that $f^k(U_i) \cap U_j \neq \emptyset$ for each $k \ge L_{i,j}$. Let L be greater than all the numbers $L_{i,j}$, with $i, j \in \{1, \dots, m\}$ and let $(\{I_1, \dots, I_n\}; P)$ be an L-spaced specification, where for each $j \in \{1, \dots, n\}$, $I_j = [a_j, b_j] \cap \mathbb{Z}$.

For every $y \in X$ we fix $U(y) \in \mathcal{U}$ such that $y \in U(y)$. Since $a_{j+1} - b_j \geq L$ it follows that there exists $y_j \in U(P(b_j))$ such that $f^{a_{j+1}-b_j}(y_j) \in U(P(a_{j+1}))$. Thus the sequence $(x_k)_{k=a_1}^{b_n}$ defined by

$$x_k = \begin{cases} P(k), & a_j \leq k < b_j, \;\; j \in \{1, \ldots, n\} \\ f^{k-b_j}(y_j), & b_j \leq k < a_{j+1}, \;\; j \in \{1, \ldots, n-1\} \\ P(b_n)), & k = b_n \end{cases}$$

is a finite δ-pseudo-orbit of f. The shadowing property assures the existence of a point $z \in X$ that ε-shadows $(x_k)_{k=a_1}^{b_n}$. It is easy to see that z also ε-shadows the specification $(\{I_1, \ldots, I_n\}; P)$. Hence f has the specification property. □

The following result is a corollary of Lemmas 5 and 6.

Corollary 3. *If a homeomorphism has the shadowing property then it is topologically mixing if and only if it has the specification property.*

The following property was introduced by M. L. Blank in [11] and further explored by Y. Zhang in [59].

Definition 10. *We say that a sequence $\{x_k\}_{k\in\mathbb{N}_0} \subset X$ is a* δ-average-pseudo-orbit *if there is an integer $N = N(\delta) \in \mathbb{N}$ such that for every $n \geq N$ and $i \geq 0$ the following holds*

$$\frac{1}{n}\sum_{k=0}^{n-1} d(f(x_{k+i})), x_{k+i+1}) < \delta.$$

We say that $\{x_k\}_{k\in\mathbb{N}_0}$ is ε-shadowed in average *if there exists $y \in X$ satisfying*

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} d(f^i(y), x_i) < \varepsilon.$$

We say that f has the average shadowing property *if for every $\varepsilon > 0$ there exists $\delta > 0$ such that every δ-average-pseudo-orbit is ε-shadowed in average.*

The following property was introduced by R. Gu and S. Y. Xia in [27].

Definition 11. *We say that a sequence $\{x_k\}_{k\in\mathbb{N}_0} \subset X$ is an* asymptotic average-pseudo-orbit *if it satisfies*

$$\frac{1}{n}\sum_{i=0}^{n-1} d(f(x_i)), x_{i+1}) \to 0, \quad n \to \infty.$$

We say that $\{x_k\}_{k\in\mathbb{N}_0}$ is asymptotically shadowed in average *if there exists $y \in X$ satisfying*

$$\frac{1}{n}\sum_{i=0}^{n-1} d(f^i(y), x_i) \to 0, \quad n \to \infty.$$

We say that f has the asymptotic average shadowing property *if every asymptotic-average-pseudo-orbit is asymptotically shadowed in average.*

The following Theorem is proved in [35] and plays an important role in the proof of Theorem 1.

Theorem 7 ([35])**.** *If f is a homeomorphism on a compact metric space with the shadowing property then the following conditions are equivalent:*

1. *f is totally transitive, i.e., f^n is transitive for every $n \in \mathbb{N}$;*
2. *f is topologically mixing;*
3. *f has the average shadowing property;*
4. *f has the asymptotic average shadowing property;*
5. *f has the specification property.*

The equivalence of (2) and (5) was proved in Corollary 3. We will not prove the equivalences of the average shadowing and asymptotic average shadowing properties and send the reader to the original work. We prove the equivalence between (1) and (2) in the following lemma.

Lemma 7 ([36])**.** *If a homeomorphism f has the shadowing property then it is totally transitive if and only if it is topologically mixing.*

Proof. We first note that any topologically mixing homeomorphism is totally transitive. It is easily checked that every topologically mixing homeomorphism is transitive and that if f is topologically mixing then f^n is topologically mixing for every $n \in \mathbb{N}$. The converse only holds under the assumption of the shadowing property.

We claim that for any three non-empty open sets $U_1, U_2, V \subset X$ there exists a number $K \in \mathbb{N}$ such that

$$f^K(U_1) \cap V \neq \emptyset \quad \text{and} \quad f^K(U_2) \cap V \neq \emptyset.$$

Indeed, since f is transitive there exists $L \in \mathbb{N}$ such that $f^L(U_1) \cap V \neq \emptyset$. Set

$$W = U_1 \cap f^{-L}(V)$$

and consider any point $u \in W$. Let $\varepsilon > 0$ be such that the ball centered at u and radius 3ε, $B(u, 3\varepsilon)$, is contained in W and choose $0 < \delta < \varepsilon$ such that every δ-pseudo-orbit is ε-shadowed. The transitivity of f assures the existence of a point $x \in B(u, \delta)$ and a number $N \in \mathbb{N}$ such that $f^N(x) \in B(u, \delta)$. Thus the sequence $(x_k)_{k\in\mathbb{Z}}$ defined by

$$x_k = f^{k \bmod N}(x), \quad k \in \mathbb{Z}$$

is a periodic δ-pseudo-orbit for f. Hence there exists $z \in X$ that ε-shadows it. In particular, the f^N-orbit of z is contained in $B(u, 2\varepsilon)$. Therefore the ω-limit set $\omega(z, f^N)$ of z for f^N is contained in $B(u, 3\varepsilon) \subset W$. Note that $f^N(\omega(z, f^N)) = \omega(z, f^N)$, so we proved the existence of a periodic set[1] $A = \omega(z, f^N) \subset W$. It follows that

$$f^{jN}(W) \cap W \neq \emptyset, \quad j \in \mathbb{Z},$$

and (by definition of W) that

$$f^{L+jN}(U_1) \cap V \neq \emptyset, \quad j \in \mathbb{Z}.$$

Since f is totally transitive the map f^N is transitive. So there exists $J \in \mathbb{N}$ such that $f^{JN}(U_2) \cap f^{-L}(V) \neq \emptyset$. Hence

$$f^{L+JN}(U_1) \cap V \neq \emptyset \quad \text{and} \quad f^{L+JN}(U_2) \cap V \neq \emptyset$$

and the claim is proved.

Using the claim one can prove that for any finite family of open subsets $U_1, \ldots, U_n, V \subset X$ there exists a number $K \in \mathbb{N}$ such that

$$f^K(U_i) \cap V \neq \emptyset, \quad i \in \{1, \ldots, n\}.$$

For more details see Proposition II.3 in [23]. Now we use this fact to prove that f is topologically mixing. For any pair of non-empty open subsets $U, V \subset X$ we consider points $u \in U$, $v \in V$ and any number $\varepsilon > 0$ such that $B(u, \varepsilon) \subset U$ and $B(v, \varepsilon) \subset V$. The shadowing property assures the existence of $0 < \delta < \varepsilon$ such that every δ-pseudo-orbit is ε-shadowed. We consider a periodic set $A = f^N(A) \subset B(u, \frac{\delta}{2})$, as previously, so that for any $a \in A$ the sequence $x_k = f^{k \mod N}(a), k \in \mathbb{Z}$ is a periodic δ-pseudo-orbit.

For any $i \in \{0, \ldots, N-1\}$ let $U_i = B(x_i, \delta)$ and use the fact above to the neighborhoods $U_0, \ldots, U_{N-1}, B(v, \varepsilon)$ to obtain a number $K \in \mathbb{N}$ such that

$$f^K(U_i) \cap B(v, \varepsilon) \neq \emptyset, \quad i \in \{0, \ldots, N-1\}.$$

So for each $i \in \{0, \ldots, N-1\}$ there exist a point $z_i \in B(x, \frac{\delta}{2})$ such that $f^{i+K}(z_i) \in B(v, \varepsilon)$. Thus for each $l \geq K$ we write $l = K + nN + r$, where $n \geq 0$ and $0 \leq r \leq N-1$, and consider the finite δ-pseudo-orbit $(y_k)_{k=0}^l$ defined by

$$y_k = \begin{cases} x_k, & 0 \leq k \leq nN - 1 \\ f^{k-nN}(z_r), & nN \leq k \leq l. \end{cases}$$

The point $z \in X$ that ε-shadows it belongs to U and satisfies $f^l(z) \in V$. Since this argument can be done for any pair U, V of non-empty open subsets of X this proves that f is topologically mixing and finishes the proof. □

We proved (Theorem 5) that the two-sided limit shadowing property implies shadowing and transitivity, so Theorem 7 reduces the proof of Theorem 1 to the following lemma:

[1] A periodic set is a set $A \subset X$ satisfying $f^K(A) = A$ for some $K \in \mathbb{N}$.

Lemma 8. *If a homeomorphism $f\colon X \to X$ has the two-sided limit shadowing property, then f^n also has the two-sided limit shadowing property for every $n \in \mathbb{Z} \setminus \{0\}$.*

Indeed, the two-sided limit shadowing property implies transitivity so this lemma implies f is totally transitive. Since it also implies shadowing, Theorem 7 concludes the proof.

Proof of Lemma 8: Let $n \in \mathbb{Z} \setminus \{0\}$ and $(x_k)_{k\in\mathbb{Z}}$ be a two-sided limit pseudo-orbit for f^n. The sequence $(y_k)_{k\in\mathbb{Z}}$ defined by

$$y_k = \begin{cases} x_k, & \text{if } k = in, \ \ i \in \mathbb{Z} \\ f^{k-in}(x_{in}), & \text{if } in < k < (i+1)n, \ \ i \in \mathbb{Z} \end{cases}$$

is a two-sided limit pseudo-orbit for f. Hence it is two-sided limit shadowed by a point $z \in X$. Therefore the sequence $(x_k)_{k\in\mathbb{Z}}$ is two-sided limit shadowed by the f^n-orbit of z. □

Remark 2. *As a consequence of Theorem* 1 *we know that any homeomorphism with the two-sided limit shadowing property has all features of systems with the specification property, and these are numerous* (*see Chapter* 21 *of* [*18*] *for details*)*. For example by Proposition* 21.6 *in* [*18*] *we obtain that it has positive topological entropy. We note that this also implies that there are no homeomorphisms with the two-sided limit shadowing property on the unit interval or the unit circle since there are no homeomorphism with positive topological entropy on them* (*see* [*58*] *Lemmas* 7.14 *and* 7.14.1).

Remark 3. *In Example* 2.2 *we exhibit a homeomorphism f admitting the two-sided limit shadowing property with gap* 1 *but without the two-sided limit shadowing property. Since f^2 is the identity map on $X = \{a, b\}$ it is obvious that it does not admit the two-sided limit shadowing property with any gap, so Lemma 8 does not hold if we exchange the two-sided limit shadowing property for the two-sided limit shadowing with a gap.*

The last results of this section generalize Lemma 7 when X is a path connected metric space or X is a compact connected metric space.

Proposition 2. *If a homeomorphism on a path connected metric space (not necessarily compact) has the shadowing property then it is transitive if and only if it is topologically mixing.*

Proof. It is enough to prove that f is totally transitive. Since it has the shadowing property it is enough to prove that f is totally chain-transitive, that is, f^n is chain-transitive for every $n \in \mathbb{N}$. If f is a transitive homeomorphism then, in particular, the non-wandering set[2] $\Omega(f)$ equals to the whole space X. It is an interesting exercise to prove that $\Omega(f^n) = X$ for every $n \in \mathbb{N}$ (one can also see [44] Proposition 3.3). The proposition follows from the following fact: if g is a homeomorphism on a path connected metric space X such that $\Omega(g) = X$ then g is chain-transitive[3].

[2]A point $p \in M$ is said to be *non-wandering* if it satisfies: for each neighborhood U of p there exists $n \in \mathbb{N}$ such that $f^n(U) \cap U \neq \emptyset$. The set of all non-wandering points is called the non-wandering set and is usually denoted by $\Omega(f)$.

[3]I learned this fact in a lecture of my friend Davi Obata at IMPA

To prove this, let $x, y \in X$ and consider a curve $\gamma\colon [0,1] \to X$ such that $\gamma(0) = g(x)$ and $\gamma(1) = y$. The fact that all points of this curve are non-wandering points allows us to create finite ε-pseudo-orbits connecting x to y for every $\varepsilon > 0$. Indeed, consider a partition $0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = 1$ of $[0,1]$ such that

$$d(\gamma(t_i), \gamma(t_{i+1})) < \frac{\varepsilon}{3}, \quad i \in \{0, \ldots, n-1\}$$

and choose points $y_i \in B(\gamma(t_i), \frac{\varepsilon}{3})$ and numbers $k_i \in \mathbb{N}$ such that $g^{k_i}(y_i) \in B(\gamma(t_i), \frac{\varepsilon}{3})$. If we let $N_0 = 0$ and $N_i = \sum_{j=0}^{i} k_j$ for $i \in \{1, \ldots, n-1\}$ then the sequence $(x_k)_{k=0}^{N_{n-1}}$ defined by

$$x_k = \begin{cases} x, & k = 0 \\ g^{k-1}(y_0), & 1 \leq k \leq k_0 \\ g^{k-N_i}(y_i), & N_i \leq k < N_{i+1}, \ \ i \in \{1, \ldots, n-1\} \\ y, & k = N_{n-1} \end{cases}$$

is a finite ε-pseudo-orbit connecting x to y. This proves the fact and also the proposition. □

Proposition 3. *If a homeomorphism on a compact and connected metric space has the shadowing property then it is transitive if and only if it is topologically mixing.*

Proof. By Proposition 7 it is enough to prove that f is totally transitive. If f is a transitive homeomorphism in a compact metric space such that f^n is not transitive, $n > 1$, then f admits a periodic decomposition of length n, which means that there exist disjoint compact subsets $W_0, \ldots, W_{n-1} \subset X$ such that $X = \bigcup_{i=1}^{n} W_i$, $f(W_i) = W_{i+1 \bmod n}$ for every $i \in \{0, \ldots, n-1\}$, and f^n restricted to each W_i is topologically mixing. Since X is connected it follows that $n = 1$, which is a contradiction. This proves that f is totally transitive and finishes the proof. □

In view of all results of this section the following question seems natural:

Question 2. *Does every homeomorphism with shadowing and specification properties have the two-sided limit shadowing property? And the limit shadowing property?*

We think the answer for this question is no because the two-sided limit shadowing property has some connections with expansiveness as we will see in the next chapter.

Chapter 3

Two-sided limit shadowing and expansiveness

In this chapter we characterize the two-sided limit shadowing property and the two-sided limit shadowing property with a gap for expansive homeomorphisms. In this case the two-sided limit shadowing property with a gap is equivalent to shadowing and transitivity, while the two-sided limit shadowing property is equivalent to shadowing and topological mixing. At the end we exhibit some systems with the two-sided limit shadowing property outside the class of expansive homeomorphisms.

3.1 Two-sided limit shadowing for expansive homeomorphisms

We begin with the definition of an expansive homeomorphism.

Definition 12. *A homeomorphism* $f\colon X \to X$ *is* expansive *if there exists a number* $c > 0$ *such that if* $x, y \in X$ *satisfy* $d(f^n(x), f^n(y)) \leq c$ *for every* $n \in \mathbb{Z}$ *then* $x = y$*.*

This property says that two different orbits move away from each other, either in the future or in the past. It was extensively studied and is still the center of many researches. We refer the reader to [7] which contains some important results about this class of systems. Our first result answers Question 2 assuming f is expansive.

Proposition 4 ([15])**.** *Every expansive homeomorphism with the shadowing and specification properties has the two-sided limit shadowing property.*

Before proving this proposition we state some definitions and results.

Definition 13. *We define the* stable set *of $y \in X$ as the set of points that limit shadows the future orbit of y*

$$W^s(y) = \{x \in X;\ d(f^k(x), f^k(y)) \to 0,\ \text{if } k \to \infty\}.$$

We define the unstable set *of y as the set of points that limit shadows in the past the past orbit of y*

$$W^u(y) = \{x \in X;\ d(f^{-k}(x), f^{-k}(y)) \to 0,\ \text{if } k \to \infty\}.$$

Remark 4. *It is obvious that if $y \in W^s(x)$ then $x \in W^s(y)$ and the ω-limit sets of x and y are the same. Also, if $z \in W^u(x)$ then $x \in W^u(z)$ and the α-limit sets of x and z are the same.*

Definition 14. *For some number $\varepsilon > 0$ we define the ε-*stable set *of $y \in X$ as the set of points that ε-shadows the future orbit of y*

$$W^s_\varepsilon(y) = \{x \in X;\ d(f^k(x), f^k(y)) < \varepsilon \text{ for every } k \in \mathbb{N}\}.$$

*We define the ε-*unstable set *of y as the set of points that ε-shadows the past orbit of y*

$$W^u_\varepsilon(y) = \{x \in X;\ d(f^{-k}(x), f^{-k}(y)) < \varepsilon \text{ for every } k \in \mathbb{N}\}.$$

An important fact about expansive homeomorphisms is that there exists a number $\varepsilon > 0$ such that for every $y \in X$ the following inclusions hold (see Lemma 1 in [42])

$$W^s_\varepsilon(y) \subset W^s(y) \quad \text{and} \quad W^u_\varepsilon(y) \subset W^u(y).$$

This says that if the future orbits (or past orbits) of two points in X remain ε-close then they must be asymptotically close. We use this fact to prove the first lemma of this chapter following the proof of Theorem 2.1 in [19].

Lemma 9. *Every expansive homeomorphism with the shadowing property has both limit shadowing and negative limit shadowing properties.*

Proof. We begin fixing the number $\varepsilon > 0$ such that for every $y \in X$ the following inclusions hold

$$W^s_\varepsilon(y) \subset W^s(y) \quad \text{and} \quad W^u_\varepsilon(y) \subset W^u(y).$$

For each $j \in \mathbb{N}$ the shadowing property assures the existence of a number $\delta_j > 0$ such that every δ_j-pseudo-orbit is $\frac{\varepsilon}{j+1}$-shadowed. Let $(x_k)_{k\in\mathbb{N}_0}$ be a limit pseudo-orbit. For each $j \in \mathbb{N}$ choose $k_j \in \mathbb{N}$ such that for each $k \geq k_j$ the following holds

$$d(f(x_k), x_{k+1}) < \delta_j.$$

The shadowing property assures the existence of points $y_j \in X$ such that for each $k \geq k_j$ we have

$$d(f^k(y_j), x_k) < \frac{\varepsilon}{j+1}.$$

We claim that $(x_k)_{k\in\mathbb{N}_0}$ is limit shadowed by y_1. Indeed for each $j \in \mathbb{N}$ and each $k \geq k_j$ the following inequality holds

$$\begin{aligned} d(f^k(y_1), f^k(y_j)) &\leq d(f^k(y_1), x_k) + d(x_k, f^k(y_j)) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{j+1} \\ &< \varepsilon. \end{aligned}$$

Or equivalently, $f^{k_j}(y_1) \in W^s_\varepsilon(f^{k_j}(y_j))$. By the choice of ε we obtain

$$d(f^k(y_1), f^k(y_j)) \to 0, \quad k \to \infty$$

for all $j \in \mathbb{N}$. This implies that

$$d(f^k(y_1), x_k) \to 0, \quad k \to \infty$$

and proves the claim. Hence every limit pseudo-orbit is limit shadowed and f has the limit shadowing property. To prove the negative limit shadowing property one just have to note that the inverse map f^{-1} is expansive and has the shadowing property and apply the previous argument to prove that f^{-1} has the limit shadowing property, which is equivalent to the negative limit shadowing property (Proposition 1). □

Corollary 4. *A chain-transitive and expansive homeomorphism has the shadowing property if and only if it has the limit shadowing property.*

Proof. If it has the shadowing property Lemma 9 assures it has the limit shadowing property. We already proved that the limit shadowing property implies shadowing in the chain-transitive scenario (see Theorem 6). □

The following corollary answers Question 1 affirmatively in the chain-transitive expansive scenario.

Corollary 5. *A chain-transitive expansive homeomorphism has the limit shadowing property if and only if it has the negative limit shadowing property.*

Proof. We proved in Corollary 4 that under these hypothesis the limit shadowing property is equivalent to the shadowing property. Therefore, Lemma 9 implies the negative limit shadowing property. It is easy to see that f^{-1} is a chain-transitive and expansive homeomorphism, so Corollary 4 assures it has the shadowing property. This implies f has the shadowing property and Lemma 9 assures f has the limit shadowing property. □

Now we are ready to prove Proposition 4.

Proof of Proposition 4: Let $\{x_k\}_{k\in\mathbb{Z}}$ be a two-sided limit pseudo-orbit. Since f has the limit shadowing and negative limit shadowing properties (Lemma 9) there exists points $p_1, p_2 \in X$ satisfying

$$d(f^k(p_1), x_k) \to 0, \quad k \to -\infty$$

and

$$d(f^k(p_2), x_k) \to 0, \quad k \to \infty.$$

Since f has the shadowing property and the specification property, there exist $\delta > 0$ and $L \in \mathbb{N}$ such that every δ-pseudo-orbit is ε-shadowed and every L-spaced specification is δ-shadowed, where ε is chosen as above. Choose $N \in \mathbb{N}$ such that $2N \geq L$ and that for every $k \geq N$ the following holds

$$d(f^{-k}(p_1), x_{-k}) < \delta \quad \text{and} \quad d(f^k(p_2), x_k) < \delta.$$

Let $I_1 = \{-N\}$, $I_2 = \{N\}$, $P(-N) = f^{-N}(p_1)$ and $P(N) = f^N(p_2)$. Since $(\{I_1, I_2\}; P)$ is a L-spaced specification there exists a point $z \in X$ satisfying

$$d(f^{-N}(z), f^{-N}(p_1)) = d(f^{-N}(z), P(-N)) < \delta$$

and

$$d(f^N(z), f^N(p_2)) = d(f^N(z), P(N)) < \delta.$$

This implies that the sequence $(y_k)_{k\in\mathbb{Z}}$ defined by

$$y_k = \begin{cases} f^k(p_1), & k < -N \\ f^k(z), & -N \leq k \leq N \\ f^k(p_2), & k > N \end{cases}$$

is a δ-pseudo orbit. Hence there exists a point $\tilde{z} \in X$ that ε-shadows it. In particular,

$$d(f^k(\tilde{z}), f^k(p_1)) < \varepsilon, \quad k \leq -N$$

and

$$d(f^k(\tilde{z}), f^k(p_2)) < \varepsilon, \quad k \geq N.$$

By the choice of ε we obtain

$$d(f^k(\tilde{z}), f^k(p_1)) \to 0, \quad k \to -\infty$$

and

$$d(f^k(\tilde{z}), f^k(p_2)) \to 0, \quad k \to \infty.$$

Since p_1 limit-shadows in the past $(x_k)_{k\in-\mathbb{N}_0}$ and p_2 limit-shadows $(x_k)_{k\in\mathbb{N}_0}$ it follows that $\tilde{z}$ two-sided limit shadows $(x_k)_{k\in\mathbb{Z}}$. Therefore f has the two-sided limit shadowing property. □

In the transitive (mixing) scenario we obtain a more precise characterization.

Theorem 8 ([17]). *If f is a transitive and expansive homeomorphism on a compact metric space then the following are equivalent:*

1. *f has the shadowing property*

2. *f has the limit shadowing property*

3. f has the two-sided limit shadowing property with a gap.

If, in addition, f is topologically mixing then these properties are equivalent to the two-sided limit shadowing property.

Proof. It was proved in Lemma 9 that (1) implies (2) and in Theorem 5 that (3) implies (1). It only remains to prove that (2) implies (3) to obtain the first equivalences. We know that the limit shadowing property implies shadowing in the transitive scenario (see Theorem 6). Thus it remains to prove that shadowing and transitivity implies two-sided limit shadowing with a gap for expansive homeomorphisms. It follows from the Spectral Decomposition Theorem ([7, Theorem 3.1.11]) for topologically Anosov maps that there exists an integer N such that X can be written as a disjoint union, $X = B_0 \cup \ldots \cup B_{N-1}$ of non-empty closed sets satisfying

$$f(B_i) = B_{(i+1) \bmod N} \quad \text{for } i = 0, \ldots, N-1,$$

and such that $f^N|_{B_i} \colon B_i \to B_i$ is topologically mixing (hence has the specification property by Lemma 6) for each i. Note that this happens because f is transitive, so it has only one chain-recurrent class. We claim that f has the two-sided limit shadowing property with gap $N-1$. Let $(x_k)_{k\in\mathbb{Z}}$ be a two-sided limit pseudo-orbit for f. It is easy to see that there exists positive integers $L_1, L_2 \in \mathbb{N}$ such that

$$x_{L_1+n} \in B_{n \bmod N} \text{ and } x_{-L_2-n} \in B_{(-n) \bmod N} \quad \text{for each } n \geq 0.$$

Let $K = (L_1 + L_2) \bmod N$. Take any $n_0 \in \mathbb{N}$ such that $(-L_2) \bmod N + Nn_0 \geq L_1$. It follows that

$$f^K(x_{(-L_2) \bmod N+Nn}) \in B_0 \quad \text{for } n > n_0.$$

Let $p_1, \ldots, p_{n_0}$ be any points in B_0. Then the sequence $(y_k)_{k\in\mathbb{Z}}$ defined by

$$y_k = \begin{cases} x_{(-L_2 \bmod N)+Nk}, & \text{for } k \leq 0, \\ p_k, & \text{for } 1 \leq k \leq n_0, \\ f^K(x_{(-L_2) \bmod N+Nk}) & \text{for } k > n_0, \end{cases}$$

is a two-sided limit pseudo-orbit for f^N contained in B_0. By Proposition 4 f^N restricted to B_0 has the two-sided limit shadowing property, since $f^N|_{B_0}$ has both shadowing and specification. Therefore $(y_k)_{k\in\mathbb{Z}}$ is two-sided limit shadowed for f^N by a point $z \in B_0$. It implies that $f^{-K}(z)$ limit shadows for f^N the limit pseudo-orbit $(f^{-K}(y_k))_{k=0}^{\infty}$ (this is a limit pseudo-orbit with respect to f^N). For $k > n_0$ we have $x_{(-L_2) \bmod N+Nk} = f^{-K}(y_k)$. Hence the sequence $(x_k)_{k\in\mathbb{Z}}$ is two-sided limit shadowed for f with gap K by the point $z \in X$ since uniform continuity of f and two-sided limit shadowing property for f^N clearly imply

$$d(f^k(z), x_k) \to 0, \quad k \to -\infty,$$
$$d(f^{-K+k}(z), x_k) \to 0 \quad k \to \infty.$$

In the topologically mixing case the shadowing property implies the specification property (Lemma 6), so Proposition 4 together with Theorem 1 shows that the shadowing property is equivalent to the two-sided limit shadowing property. This concludes the proof. □

In particular, the following holds.

Theorem 9 ([17])**.** *Let f be an expansive homeomorphism on a compact metric space. Then*

1. *f is transitive and has the shadowing property if and only if f has the two-sided limit shadowing property with a gap;*
2. *f is topologically mixing and has the shadowing property if and only if f has the two-sided limit shadowing property.*

When X is connected, the Spectral Decomposition Theorem says that f is transitive if and only if it is topologically mixing. Hence we have the following corollary.

Theorem 10 ([17])**.** *If f is an expansive homeomorphism on a compact and connected metric space then the following are equivalent:*

1. *f is transitive and has the shadowing property;*
2. *f is topologically mixing and has the shadowing property;*
3. *f has the two-sided limit shadowing property with a gap;*
4. *f has the two-sided limit shadowing property.*

For any compact metric space (X, d) we consider $X^{\mathbb{Z}}$ the product of countable many copies of X with the Tichonov (product) topology. Points in $X^{\mathbb{Z}}$ are sequences $(x_k)_{k\in\mathbb{Z}}$ whose all coordinates belong to X. We define a map $\sigma\colon X^{\mathbb{Z}} \to X^{\mathbb{Z}}$ by

$$\sigma((x_k)_{k\in\mathbb{Z}}) = (x_{k+1})_{k\in\mathbb{Z}}.$$

This map is called the *shift map*. It is easily seen that σ is a homeomorphism on the compact space $X^{\mathbb{Z}}$. A special case of this construction is the *full r-shift* $\Omega_r = X^{\mathbb{Z}}$, where $r \in \mathbb{N}$ and $X = \Lambda_r = \{0, 1, \ldots, r-1\}$ is equipped with discrete metric.

Definition 15. *Any σ-invariant and closed subset of Ω_r is called a* shift space*. A shift space $Z \subset \Omega_r$ is a* subshift of finite type *if there exists $N \in \mathbb{N}$ and a subset $\mathcal{L}$ of $\prod_{i=0}^{N} \Lambda_r$ such that $x = (x_n)_{n\in\mathbb{Z}} \in Z$ if and only if $x_j x_{j+1} \ldots x_{j+N} \in \mathcal{L}$ for every $j \in \mathbb{Z}$.*

We obtain a variant of Walters theorem from [57] (see also [7, Theorem 2.3.18]) which characterizes shifts of finite type as the shift spaces with shadowing. Using this theorem and Theorem 9 we obtain characterizations of transitive and topologically mixing shifts of finite type in terms of two-sided limit shadowing properties.

Theorem 11 ([17])**.** *Let X be a shift space. Then*

1. *X is a transitive shift of finite type if and only if $\sigma\colon X \to X$ has the two-sided limit shadowing property with a gap;*

2. *X is topologically mixing shift of finite type if and only if $\sigma\colon X \to X$ has the two-sided limit shadowing property.*

We are finally ready to exhibit a homeomorphism with the simple two-sided limit shadowing property that does not have the two-sided limit shadowing property. The following proposition plays an important role in this direction.

Proposition 5. *If a homeomorphism on a compact metric space is expansive and has the specification property then it has the simple two-sided limit shadowing property.*

Proof. Let $(x_k)_{k\in\mathbb{Z}}$ be a simple two-sided limit pseudo-orbit. It is the past orbit of a point $x \in X$ and the future orbit of a point $y \in X$. Let $\varepsilon > 0$ be such that $W^s_\varepsilon(y) \subset W^s(y)$ and $W^u_\varepsilon(x) \subset W^u(x)$. Choose $L \in \mathbb{N}$ such that every L-spaced specification is $\frac{\varepsilon}{2}$-shadowed. For each $k \geq L$ we consider $I^1_k = [-k, 0]$, $I^2_k = [L, k]$ and $P\colon I^1_k \cup I^2_k \to X$ defined by

$$P(n) = \begin{cases} f^n(x), & n \in I^1_k, \\ f^n(y), & n \in I^2_k. \end{cases}$$

Thus for each $k \geq L$ the triple (I^1_k, I^2_k, P) is a L-spaced specification and hence there exists a point $z_k \in X$ that $\frac{\varepsilon}{2}$-shadows it. Note that $z_k \in B(x, \frac{\varepsilon}{2})$ for every $k \geq L$ and that there is a point $z \in B(x, \varepsilon)$ that is the limit of some subsequence of $(z_k)_{k\geq L}$. We will prove that z two-sided limit shadows $(x_k)_{k\in\mathbb{Z}}$. Indeed, for each $k \in \mathbb{N}$ we can choose $n_k \geq \max\{k, L\}$ such that

$$d(f^n(z), f^n(z_{n_k})) < \frac{\varepsilon}{2}, \quad -k \leq n \leq 0.$$

Thus for every $k \geq 0$ we have

$$\begin{aligned} d(f^{-k}(z), x_{-k}) &\leq d(f^{-k}(z), f^{-k}(z_{n_k})) + d(f^{-k}(z_{n_k}), x_{-k}) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon. \end{aligned}$$

By the choice of ε we obtain

$$d(f^k(z), x_k) \to 0, \quad k \to -\infty.$$

Also, for $k \geq L$ we obtain $d(f^k(z), x_k) < \varepsilon$, which implies

$$d(f^k(z), x_k) \to 0, \quad k \to \infty.$$

Therefore z two-sided limit shadows $(x_k)_{k\in\mathbb{Z}}$ and the proposition is proved. □

As a corollary we obtain the desired result.

Corollary 6. *If a two-sided shift has the specification property but is not a shift of finite type then it has the simple two-sided limit shadowing property and does not have the two-sided limit shadowing property.*

Proof. Last proposition assures it has the simple two-sided limit shadowing property and Walter's Theorem assures it does not have the shadowing property, since it is not a shift of finite type. Theorem 1 assures it does not have the two-sided limit shadowing property. □

3.2 Non-expansive homeomorphisms with two-sided limit shadowing

In this section we prove two results which will allow us to provide examples of homeomorphisms with the two-sided limit shadowing property

1. which are non-expansive;
2. without the periodic specification property.

The first result is the following.

Theorem 12 ([17])**.** *For every compact metric space X the shift map $\sigma\colon X^{\mathbb{Z}} \to X^{\mathbb{Z}}$ has the two-sided limit shadowing property.*

Proof. As two-sided limit shadowing property does not depend on the choice of metric for the space we can pick any equivalent metric for $X^{\mathbb{Z}}$. We equip $X^{\mathbb{Z}}$ with the metric D defined for $x = (x_j)_{j\in\mathbb{Z}}, y = (y_j)_{j\in\mathbb{Z}} \in X^{\mathbb{Z}}$ by

$$D(x,y) = \sup_{j\in\mathbb{Z}} \frac{d(x_j,y_j)}{2^{|j|}},$$

where d is any metric for X such that $\operatorname{diam} X \leq 1$. Let $\{x^{(n)}\}_{n\in\mathbb{Z}}$ be any two-sided limit pseudo-orbit for σ. It follows that for every $p \geq 1$ there is N_p such that $D(\sigma(x^{(m)}), x^{(m+1)}) \leq 2^{-p-1}$ for all $|m| \geq N_p$. It implies that for m as before and for each $j \in \mathbb{Z}$ we have

$$\frac{1}{2^{p+1}} \geq D(\sigma(x^{(m)}), x^{(m+1)}) \geq \frac{1}{2^{|j|}} d(x^{(m)}_{j+1}, x^{(m+1)}_j). \tag{3.1}$$

Define a point $x \in X^{\mathbb{Z}}$ by $x_j = x^{(j)}_0$ for $j \in \mathbb{Z}$, in other words

$$x = (\ldots, x^{(-1)}_0, x^{(0)}_0, x^{(1)}_0 \ldots)$$

We claim that x two-sided limit shadows $\{x^{(n)}\}_{n\in\mathbb{Z}}$. To see this, note first that

$$\sigma^m(x)_k = x^{(m+k)}_0 \quad \text{for all } k, m \in \mathbb{Z}.$$

On the other hand

$$D(\sigma^m(x), x^{(m)}) = \sup_{j\in\mathbb{Z}} \frac{1}{2^{|j|}} d(\sigma^m(x)_j, x_j^{(m)}) = \sup_{j\in\mathbb{Z}} \frac{1}{2^{|j|}} d(x_0^{(m+j)}, x_j^{(m)}).$$

By the triangle inequality for all $m \in \mathbb{Z}$ and k positive

$$d(x_0^{(m+k)}, x_k^{(m)}) \le \sum_{j=0}^{k-1} d(x_j^{(m+k-j)}, x_{j+1}^{(m+k-j-1)}) =$$
$$= d(x_0^{(m+k)}, x_1^{(m+k-1)}) + d(x_1^{(m+k-1)}, x_2^{(m+k-2)}) + \ldots + d(x_{k-1}^{(m+1)}, x_k^{(m)}).$$

Similarly, for k negative

$$d(x_0^{(m+k)}, x_k^{(m)}) \le \sum_{j=0}^{|k|-1} d(x_{-j}^{(m+k+j)}, x_{-j-1}^{(m+k+j+1)}) =$$
$$= d(x_0^{(m+k)}, x_{-1}^{(m+k+1)}) + d(x_{-1}^{(m+k+1)}, x_{-2}^{(m+k+2)}) + \ldots + d(x_{k+1}^{(m-1)}, x_k^{(m)}).$$

If $m > N_p$, $k > 0$ and $0 \le j < k$, then we may apply (3.1) and obtain

$$d(x_j^{(m+k-j)}, x_{j+1}^{(m+k-j-1)}) \le \frac{2^{|j|}}{2^{p+1}}.$$

In particular,

$$\sum_{j=0}^{k-1} d(x_j^{(m+k-j)}, x_{j+1}^{(m+k-j-1)}) \le \frac{1 + 2 + \ldots + 2^{k-1}}{2^{p+1}} \le \frac{2^k}{2^{p+1}}$$

If $m < -N_p$, $k < 0$ and $0 \le j \le |k| - 1$, then we may apply (3.1) and obtain

$$d(x_{-j}^{(m+k+j)}, x_{-j-1}^{(m+k+j+1)}) \le \frac{2^{j+1}}{2^{p+1}}.$$

In particular,

$$\sum_{j=0}^{|k|-1} d(x_{-j}^{(m+k+j)}, x_{-j-1}^{(m+k+j+1)}) \le \frac{2 + 2^2 + \ldots + 2^{|k|}}{2^{p+1}} \le \frac{2^{|k|}}{2^p}.$$

Hence $D(\sigma^m(x), x^{(m)}) \le 2^{-p}$ for $|m| \ge N_p$, and the proof is finished. □

It follows from the above theorem and Theorem 1 that for every compact metric space X the shift map $\sigma\colon X^{\mathbb{Z}} \to X^{\mathbb{Z}}$ has specification and shadowing. It is also easy to see that if X is the unit interval $[0, 1]$ then the shift map $\sigma\colon [0, 1]^{\mathbb{Z}} \to [0, 1]^{\mathbb{Z}}$ is not expansive. In general, if the topological dimension of X is positive, then no homeomorphism of $X^{\mathbb{Z}}$ can be expansive by [7, Theorem 2.3.13] (this is a corollary to a theorem of Mañe [42] (see also [7, Theorem 2.2.40]), which states that if $f\colon X \to X$ is an expansive homeomorphism of a compact metric space, then the topological dimension of X is finite). Hence we obtain the following:

Corollary 7 ([17])**.** *There exists a non-expansive homeomorphism on a compact metric space with the two-sided limit shadowing property.*

The other result that allow us to construct examples of non-expansive homeomorphisms with the two-sided limit shadowing property is the following.

Theorem 13 ([17]). *Assume that for each $n \in \mathbb{N}$ we have a compact metric space X_n and a homeomorphism $f_n \colon X_n \to X_n$ with the two-sided limit shadowing property. Let $X = \prod_{n\in\mathbb{N}} X_n$ be the product metric space and $F \colon X \to X$ be the homeomorphism given by*

$$F(x_1, x_2, x_3, \dots) = (f_1(x_1), f_2(x_2), f_3(x_3), \dots).$$

Then F has the two-sided limit shadowing property.

Proof. We equip X with the metric D defined for $x = (x_n)_{n\in\mathbb{N}}, y = (y_n)_{n\in\mathbb{N}} \in X$ by

$$D(x, y) = \sup_{n\in\mathbb{N}} \frac{d_n(x_n, y_n)}{2^n},$$

where d_n is any metric for X_n such that the diameter of X_n with respect to d_n fulfills $\operatorname{diam} X_n \leq 1$. It is then easy to see that if $(x^{(j)})_{j\in\mathbb{Z}}$ is a two-sided limit pseudo-orbit for F, then for each $n \in \mathbb{N}$ the sequence $(x_n^{(j)})_{j\in\mathbb{Z}}$ is a two-sided limit pseudo-orbit for f_n. Let y_n be a point which two-sided limit shadows $(x_n^{(j)})_{j\in\mathbb{Z}}$ with respect to f_n. It is again easy to see that $(y_1, y_2, y_3, \dots)$ two-sided limit shadows $(x^{(j)})_{j\in\mathbb{Z}}$ for F. □

As the last result of this chapter we prove the following corollary.

Corollary 8 ([17]). *There exists a homeomorphism with the two-sided limit shadowing property but without the periodic specification property.*

Let p and q be relatively prime integers and set $r = p + q - 1$. Define a shift of finite type $X_{(p,q)} \subset \Omega_r$ by specifying

$$\mathcal{L} = \{01, 12, \dots, (p-2)(p-1), (p-1)0, 0p, p(p+1), \dots, (p+q-2)(p+q-1), (p+q-1)0\}.$$

In other words, $X_{(p,q)}$ consists of sequences of vertices visited during a bi-infinite walk on the directed graph with two loops: one of length p with vertices labelled $0, \dots, p-1$ and one of length q with vertices labelled $0, p, p+1, \dots, p+q-2$. See figure **??** for an example.

Since the graph is connected and has two cycles with relatively prime lengths it presents a topologically mixing shift of finite type (for details see [40]). Moreover, this shift of finite type does not have any periodic point with primary period smaller than $\min\{p, q\}$. Let $(p_j)_{j=1}^{\infty}$ be a strictly increasing sequence of prime numbers. Let $X_n = X_{(p_n, p_{n+1})}$ for $n \in \mathbb{N}$, and σ_n be a shift transformation on $\Omega_{p_n+p_{n+1}-1}$ restricted to X_n. This family fulfills the assumptions of Theorem 13 so the product system has the two-sided limit shadowing property. It is easy to see that the product system $F = \sigma_1 \times \sigma_2 \times \dots$ on $X = \prod_{n=1}^{\infty} X_n$ also has the specification property, but this specification property is not periodic, since there are no periodic points for F in X.

We note that examples constructed in this section are not defined on manifolds. So the following question is natural:

Question 3. *What is the relation between the two-sided limit shadowing property and expansiveness for homeomorphisms defined on manifolds?*

Actually, we think the following is true:

Conjecture 1. *Any homeomorphism on the torus* $\mathbb{T}^n$ *with the two-sided limit shadowing property is expansive.*

Chapter 4

Anosov diffeomorphisms and the two-sided limit shadowing property

All results above allowed us to obtain characterizations of two well known conjectures on Anosov diffeomorphisms in terms of the two-sided limit shadowing property. We also give some ideas on how to solve these conjectures.

4.1 Anosov diffeomorphisms

An *Anosov diffeomorphism* on a smooth manifold M is a smooth diffeomorphism $f\colon M \to M$ satisfying:

1. for every $x \in M$ there is a splitting $T_xM = E^s(x) \oplus E^u(x)$ which is invariant under the derivative map $Df(x)\colon T_xM \to T_{f(x)}M$, that is,

$$Df(x)(E^s(x)) = E^s(f(x)) \text{ and } Df(x)(E^u(x)) = E^u(f(x)).$$

   We call $E^s(x)$ the *stable space* of x and $E^u(x)$ is called the *unstable space* of x.

2. there exist a Riemannian metric on M and a constant $0 < \lambda < 1$ such that

$$|Df^k(x)(v)|_{f^k(x)} \leq \lambda^k |v|_x \text{ and } |Df^{-k}(x)(w)|_{f^{-k}(x)} \leq \lambda^k |w|_x$$

   for all $v \in E^s(x)$, $w \in E^u(x)$, $k \in \mathbb{Z}$ and $x \in M$, where $|.|_x$ denotes the norm in T_xM induced by the Riemannian metric. This metric is said to be *adapted* to f.

Such systems have been intensely studied since the works of Anosov [6] and Smale [56] in the sixties. They introduced several examples of Anosov diffeomorphisms and stated some questions about them that has been not answered yet (to my best knowledge). The central problem on this theory is to understand all examples of Anosov diffeomorphisms (up to topological conjugacy). Smale conjectured that every Anosov diffeomorphism must be topologically conjugated to an Anosov automorphism on an infra-nilmanifold. The following properties are expected to be true:

If $f : M \to M$ is an Anosov diffeomorphism then

1. M is an infra-nilmanifold and the universal covering is the Euclidean space $\mathbb{R}^n$,
2. the lift of f to $\mathbb{R}^n$ is topologically conjugated to a hyperbolic matrix,
3. the stable and unstable foliations are on global product structure,
4. f admits a fixed point,
5. if M is compact and connected then f is transitive.

We recall some definitions. We denote by M a closed and connected smooth n-dimensional manifold, by $\widetilde{M}$ its universal covering and by $\pi\colon \widetilde{M} \to M$ the covering projection. Consider a Riemannian metric $\langle .,. \rangle$ in M and the distance d induced by this metric. We can lift this metric to a Riemannian metric in $\widetilde{M}$ (which we also denote by $\langle .,. \rangle$) as follows: for $x \in \pi^{-1}(x_0)$ and $v, w \in T_x\widetilde{M}$ we define

$$\langle v, w \rangle_x = \langle D\pi(x)(v), D\pi(x)(w) \rangle_{x_0} ,$$

where $\langle .,. \rangle_{x_0}$ is the inner product in $T_{x_0}M$. We also denote by d the distance in $\widetilde{M}$ induced by the lifted metric. By definition the covering map π is a local isometry. Hence there exists $\varepsilon_0 > 0$ such that for each $x \in \widetilde{M}$, π maps the ε_0-neighborhood of x isometrically onto the ε_0-neighborhood of $\pi(x)$.

For an Anosov diffeomorphism $f_0\colon M \to M$ we consider $f\colon \widetilde{M} \to \widetilde{M}$ any lift of f_0 to the universal covering. Since π is a local diffeomorphism, the derivative map $D\pi(x)\colon T_x\widetilde{M} \to T_{x_0}M$ is a linear isomorphism. So the splitting $T_{x_0}M = E^s(x_0) \oplus E^u(x_0)$ can be lifted to a splitting $T_x\widetilde{M} = E^s(x) \oplus E^u(x)$ that is invariant by $Df(x)$. If the adapted metric is lifted then it is easy to check that f is an Anosov diffeomorphism and that the lifted metric is adapted to f.

The stable set $W^s(x_0)$ and the unstable set $W^u(x_0)$ of an Anosov diffeomorphism are leaves of two respective foliations which we call *stable foliation* and *unstable foliation*. We denote by $\widetilde{W}^s(x_0)$ and by $\widetilde{W}^u(x_0)$ the lift of the stable and unstable leaves, respectively, to the universal covering. Actually, $\widetilde{W}^s(x_0)$ ($\widetilde{W}^u(x_0)$) is the stable (unstable) set of x with respect to the lifted Anosov diffeomorphism.

Definition 16. *The stable and the unstable foliations are on* global product structure *if for every $x, y \in M$ the leaves $\widetilde{W}^s(x)$ and $\widetilde{W}^u(y)$ intersect in exactly one point in the universal covering. If this is the case, we say that f_0 is a* product Anosov diffeomorphism*.*

Product Anosov diffeomorphisms are transitive but the converse of this statement is not known.

Remark 5. *An Anosov diffeomorphism is a product Anosov diffeomorphism if and only if its lift to the universal covering has the unique simple two-sided limit shadowing property. More generally, an Anosov diffeomorphism satisfies $W^u(x) \cap W^s(y) \neq \emptyset$ for every $x, y \in M$ if and only if it has the simple two-sided limit shadowing property.*

It is known that Anosov diffeomorphisms have the shadowing property and are expansive (see [53]). Also, on a connected and compact manifold transitive Anosov diffeomorphisms are topologically mixing and thus have the specification property (Lemma 6). So transitive Anosov diffeomorphisms satisfy all hypotheses of Proposition 4 and we obtain the following:

Theorem 14 ([15])**.** *Transitive Anosov diffeomorphisms on a compact and connected manifold have the two-sided limit shadowing property.*

This together with Theorem 1 proves the first assertion of Theorem 2.

4.2 Limit shadowing on the universal covering

In [31] (and also in [34]) it is proved that a homeomorphism $f\colon M \to M$ has the *shadowing property* if and only if any lift of f to the universal covering also has it. We prove this also holds for the *limit shadowing property.*

Lemma 10. *If $f\colon M \to M$ is a homeomorphism and $\tilde{f}\colon \widetilde{M} \to \widetilde{M}$ is any lift of f to the universal covering then $\tilde{f}$ has the limit shadowing property if and only if f also has it.*

Proof. Suppose that $\tilde{f}$ has the limit shadowing property and consider $(x_k)_{k\in\mathbb{N}} \subset M$ a limit pseudo-orbit for f. Choose $N \in \mathbb{N}$ such that

$$d(f(x_k), x_{k+1}) < \varepsilon_0, \quad k \geq N,$$

where ε_0 was defined at the beginning of last section. Note that from the definition of ε_0 we have

$$\varepsilon_0 < \min\{d(\tilde{x}, \tilde{y}); x \in M, \tilde{x}, \tilde{y} \in \pi^{-1}(x), \tilde{x} \neq \tilde{y}\}.$$

Thus for each choice of $y_N \in \pi^{-1}(x_N)$ there exists a unique limit pseudo-orbit $(y_k)_{k\geq N}$ of $\tilde{f}$ such that $y_k \in \pi^{-1}(x_k)$ and $d(\tilde{f}(y_k), y_{k+1}) < \varepsilon_0$ for every $k \geq N$. Since $\tilde{f}$ has limit shadowing there exists $z \in \widetilde{M}$ that limit shadows $(y_k)_{k\geq N}$. Therefore $\pi(\tilde{f}^{-N}(z))$ limit shadows $(x_k)_{k\in\mathbb{N}}$. Since this holds for every limit pseudo-orbit it follows that f has the limit shadowing property.

Now suppose that f has the limit shadowing property and consider $(x_k)_{k\in\mathbb{N}} \subset \widetilde{M}$ a limit pseudo-orbit for $\tilde{f}$. The sequence $(\pi(x_k))_{k\in\mathbb{N}} \subset M$ is a limit pseudo-orbit for f and thus it is limit shadowed by $z \in M$. Choose $K \in \mathbb{N}$ such that

$$d(f^k(z), \pi(x_k)) < \varepsilon_0, \quad k \geq K.$$

There is a unique point $\tilde{z} \in \pi^{-1}(f^K(z))$ such that $d(\tilde{z}, x_K) < \varepsilon_0$. It is easy to check then that $\tilde{f}^{-K}(\tilde{z})$ limit shadows $(x_k)_{k\in\mathbb{N}}$. Since this holds for every limit pseudo-orbit it follows that $\tilde{f}$ has the limit shadowing property. □

We obtain Theorem 2 as a consequence of previous lemma and Lemma 1. Actually, a slightly different version of Lemma 1.

Lemma 11. *A homeomorphism on a metric space has the unique two-sided limit shadowing property if and only if f has the limit shadowing property, the negative limit shadowing property and the unique simple two-sided limit shadowing property.*

Proof of Theorem 2: It only remains to prove second assertion. As observed above any Anosov diffeomorphism has the shadowing property and is expansive. Lemma 9 assures it has the limit shadowing property. The inverse is also an Anosov diffeomorphism and also has the limit shadowing property. By Lemma 10 any lift has limit shadowing and negative limit shadowing properties. Then Remark 5 and Lemma 11 finish the proof. □

4.3 Two-sided limit shadowing on the universal covering

This section is motivated by the following question.

Question 4. *Is it true that a homeomorphism has the two-sided limit shadowing property if and only if any lift to the universal covering also has the two-sided limit shadowing property?*

One can easily see that the argument in Lemma 10 does not follow in the two-sided limit shadowing case. This question is not supposed to be true in all cases, but it is when you consider an Anosov diffeomorphism. So the following questions are natural.

Question 5. *Is it true that any lift of an Anosov diffeomorphism to the universal covering has the two-sided limit shadowing property? Is this true if we suppose that the Anosov diffeomorphism has the two-sided limit shadowing property?*

Remark 6. *The first attempt to answer Question 5 would be to use Proposition 4 that proves the two-sided limit shadowing property holds when the specification property is present. We note that the specification property does not lift to the universal covering though, it only makes sense on the compact scenario.*

On this section we exhibit some ideas that we hope will answer Question 5. We prove Theorem 3. From now on the two-sided limit pseudo-orbit $(x_k)_{k\in\mathbb{Z}}$ on the universal covering is fixed. We denote by C the set of all bilateral sequences $\bar{v} = (v_k)_{k\in\mathbb{Z}}$ where $v_k \in T_{x_k}\widetilde{M}$ for each $k \in \mathbb{Z}$. Let B denote the subset of C consisting of bounded sequences, i.e., sequences $(v_k)_{k\in\mathbb{Z}} \in C$ that satisfy

$$\sup_{k\in\mathbb{Z}} |v_k|_{x_k} < \infty,$$

where $|.|_{x_k}$ is the norm in $T_{x_k}\widetilde{M}$ induced by the lifted metric. The map $||.||\colon B \to \mathbb{R}^+$ defined by

$$||\bar{v}|| = \sup_{k\in\mathbb{Z}} |v_k|_{x_k}$$

is a norm in B that makes $(B; ||.||)$ a Banach space. We consider the subspace C_0 of B as the space of sequences $(v_k)_{k\in\mathbb{Z}} \in B$ that satisfy

$$|v_k|_{x_k} \to 0, \ \ |k| \to \infty.$$

It is easy to see that C_0 is a closed subspace of B with respect to the norm defined above, so it is also a Banach space. We define a map $F\colon C_0 \to C_0$ as follows: for each sequence $\bar{v} = (v_k)_{k\in\mathbb{Z}} \in C_0$ we define $F(\bar{v})$ as the sequence

$$F(\bar{v})_k = \exp_{x_k}^{-1} \circ f \circ \exp_{x_{k-1}}(v_{k-1}), \quad k \in \mathbb{Z},$$

where $\exp_x$ denotes the *exponential map* of $\widetilde{M}$ in the point x.

Remark 7. *If we suppose that the ambient manifold has non-positive sectional curvature then the lifted metric has no conjugated points and the exponential map* $\exp_x\colon T_x\widetilde{M} \to \widetilde{M}$ *is a global diffeomorphism for every* $x \in \widetilde{M}$. *In this case* F *is well defined. This puts a restriction on the ambient manifold but it seems to be no problem to us since it is expected that the universal covering of a manifold supporting an Anosov diffeomorphism is the Euclidean space.*

Lemma 12. *If* $\bar{v} \in C_0$ *then* $F(\bar{v}) \in C_0$.

Proof. A standard compactness argument (which we omit here) proves that f is uniformly continuous on $\widetilde{M}$. Thus for each $\varepsilon > 0$ we can choose $0 < \delta < \frac{\varepsilon}{2}$ such that $d(x, y) < \delta$ implies $d(f(x), f(y)) < \frac{\varepsilon}{2}$. Since $\bar{v} \in C_0$ and $(x_k)_{k\in\mathbb{Z}}$ is a two-sided limit pseudo-orbit of f we can choose $K \in \mathbb{N}$ such that for $|k| \geq K$ we have

$$|v_k|_{x_k} < \delta \quad \text{and} \quad d(f(x_k), x_{k+1}) < \delta.$$

Thus for $|k| > K$ we have

$$d(\exp_{x_{k-1}}(v_{k-1}), x_{k-1}) = |v_{k-1}|_{x_{k-1}} < \delta$$

which imply

$$\begin{aligned}
|F(\bar{v})_k|_{x_k} &= |\exp_{x_k}^{-1} \circ f \circ \exp_{x_{k-1}}(v_{k-1})|_{x_k} \\
&= d(f(\exp_{x_{k-1}}(v_{k-1})), x_k) \\
&\leq d(f(\exp_{x_{k-1}}(v_{k-1})), f(x_{k-1})) + d(f(x_{k-1}), x_k) \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&= \varepsilon.
\end{aligned}$$

This is enough to prove that $F(\bar{v}) \in C_0$. □

Theorem 15 ([16])**.** *There exists a bijection between the set of fixed points of F and the set of points that two-sided limit shadows $(x_k)_{k\in\mathbb{Z}}$.*

Proof. For a two-sided limit pseudo-orbit $(x_k)_{k\in\mathbb{Z}}$ of f suppose that $\bar{v}$ is a fixed point of F. Then the sequence $(\exp_{x_k}(v_k))_{k\in\mathbb{Z}}$ is an orbit that two-sided limit shadows $(x_k)_{k\in\mathbb{Z}}$. Indeed, for each $k \in \mathbb{Z}$ we have

$$v_k = F(\bar{v})_k = \exp_{x_k}^{-1} \circ f \circ \exp_{x_{k-1}}(v_{k-1}),$$

which implies

$$\exp_{x_k}(v_k) = f \circ \exp_{x_{k-1}}(v_{k-1}).$$

By induction we obtain

$$\exp_{x_k}(v_k) = f^k(\exp_{x_0}(v_0)), \quad k \in \mathbb{Z}.$$

Therefore,

$$d(f^k(\exp_{x_0}(v_0)), x_k) = d(\exp_{x_k}(v_k), x_k) = |v_k|_{x_k} \to 0, \quad |k| \to \infty,$$

that is, $\exp_{x_0}(v_0)$ two-sided limit shadows $(x_k)_{k\in\mathbb{Z}}$.

Now suppose that z two-sided limit shadows $(x_k)_{k\in\mathbb{Z}}$. For each $k \in \mathbb{Z}$ let

$$v_k = \exp_{x_k}^{-1}(f^k(z)).$$

We have $\bar{v} = (v_k)_{k\in\mathbb{Z}} \in C_0$ since

$$|v_k|_{x_k} = d(\exp_{x_k}(v_k), x_k) = d(f^k(z), x_k) \to 0, \quad |k| \to \infty.$$

Moreover, for each $k \in \mathbb{Z}$

$$F(\bar{v})_k = \exp_{x_k}^{-1} \circ f \circ \exp_{x_{k-1}}(v_{k-1}) = \exp_{x_k}^{-1}(f^k(z)) = v_k,$$

which proves $\bar{v}$ is a fixed point of F. These arguments construct the desired bijection. □

Remark 8. *It is important to note that the Banach space C_0 and also the map F depend on the two-sided limit pseudo-orbit $(x_k)_{k\in\mathbb{Z}}$.*

Remark 9. *The map F can be defined for any homeomorphism $f_0 \colon M \to M$, any lift $f \colon \widetilde{M} \to \widetilde{M}$ and any two-sided limit pseudo-orbit $(x_k)_{k\in\mathbb{Z}}$ of f in $\widetilde{M}$. It is not expected that F admit fixed points in all cases though but it is when f_0 is an Anosov diffeomorphism.*

Now we use the hyperbolic structure of an Anosov diffeomorphism to define a new map G in C_0 with more structure than F but with the same fixed points. For each $k \in \mathbb{Z}$ consider a linear isomorphism $I_k \colon T_{f(x_{k-1})}\widetilde{M} \to T_{x_k}\widetilde{M}$ satisfying

1. $I_k(E^s(f(x_{k-1}))) = E^s(x_k)$,
2. $I_k(E^u(f(x_{k-1}))) = E^u(x_k)$,

3. $|I_k(v)|_{x_k} \leq |v|_{f(x_{k-1})}$.

We note Define a map $T\colon C_0 \to C_0$ by

$$T(\bar{v})_k = I_k \circ Df(x_{k-1})(v_{k-1}), \quad k \in \mathbb{Z}.$$

Let Id denote the identity map in C_0. We prove the following.

Theorem 16 ([16])**.** *The map $Id - T$ is a bounded linear isomorphism of C_0 with bounded inverse $(Id - T)^{-1}$.*

Using this theorem we can define the map $G\colon C_0 \to C_0$ by

$$G(\bar{v}) = (Id - T)^{-1} \circ (F - T)(\bar{v}).$$

By definition F and G have the same fixed points in C_0, so Theorem 15 is enough to prove Theorem 3.

The map $(Id-T)^{-1}$ will be defined in the proof of Theorem 16. The map $F-T$ is the following:

$$(F - T)(\bar{v})_k = \exp_{x_k}^{-1} \circ f \circ \exp_{x_{k-1}}(v_{k-1}) - I_k \circ Df(x_{k-1})(v_{k-1}), \quad k \in \mathbb{Z}.$$

We do not know how to obtain fixed points for this map in the general case but we do when f is linear or when the numbers $d(f(x_k), x_{k+1})$ are sufficiently small. In the first case G is a constant map and in the second case G is a contraction in an invariant small neighborhood of $\bar{0}$ in C_0 (see [49]). We hope some fixed point theorem applies in the general case.

Since the two-sided limit shadowing property is equivalent to the simple two-sided limit shadowing property in the Anosov case (Lemma 1) it is no restriction to suppose $(x_k)_{k\in\mathbb{Z}}$ is a simple two-sided limit pseudo-orbit. In this case we may choose I_k as the identity map for $k \neq 0$. If we consider a curve $\gamma\colon [0,1] \to \widetilde{M}$ such that $\gamma(0) = x_0$ and $\gamma(1) = f(x_{-1})$ then for each $t \in [0,1]$ we can consider the simple two-sided limit pseudo-orbit $(x_k^t)_{k\in\mathbb{Z}}$ defined as the past orbit of $\gamma(t)$ and the future orbit of x_0. This induces one-parameter families of Banach spaces $(C_0^t)_{t\in[0,1]}$ and maps $(G_t)_{t\in[0,1]}$ where C_0^t and G_t were defined above with respect to $(x_k)_{k\in\mathbb{Z}}^t$.

When M has zero sectional curvature the uniqueness of space forms says that the universal covering is the Euclidean space $\mathbb{R}^n$ and the lifted metric is the Euclidean metric. In this case, the Banach space C_0 does not depend on the two-sided limit pseudo-orbit. Thus the family $(G_t)_{t\in[0,1]}$ is a one-parameter family of maps defined on the same space C_0. For $t = 0$ the sequence $(x_k^0)_{k\in\mathbb{Z}}$ is the orbit of x_0 and $\bar{0}$ is a fixed point of G_0. We think some *fixed point continuation theorem* applies and the fixed point of G_0 can be carried through $(G_t)_{t\in[0,1]}$ for a fixed point of G_1. More precisely, we would like to obtain a curve $\Gamma\colon [0,1] \to C_0$ such that $\Gamma(0) = \bar{0}$ and $\Gamma(t)$ is a fixed point of G_t for each $t \in [0,1]$.

In J. Franks PhD thesis [21] there is a similar discussion about product Anosov diffeomorphisms: it might happen (though it is not expected) that there exists some $t_0 \in (0,1)$ such that for $s < t_0$

there exists an intersection between $W^u(\gamma(s))$ and $W^s(x_0)$ but for $s \geq t_0$ there is not. In this case the intersections $W^u(\gamma(s)) \cap W^s(x_0)$ go to infinity in $\mathbb{R}^n$ when s converges to t_0 to the left. In our setting this means the set

$$K = \{\bar{v} \in C_0;\ \ \bar{v} = G_t(\bar{v})\ \text{ for some }\ t \in [0,1]\}$$

is unbounded in C_0. The problem is to determine if this set is bounded or not. If we could prove that K is bounded then it would follow that there exists a finite number of points in $W^u(x) \cap W^s(y)$, which is a weaker form of product structure.

Remark 10. *The only fixed points continuations theorems we know are for completely continuous maps on Banach spaces. These maps satisfy that the image of every bounded set is compact. Moreover, there is some hypothesis on the set of solutions that is called* a priori bound hypothesis *and is usually the most difficult one to check (see [38]). This hypothesis is equivalent to the set K defined above be bounded.*

Question 6. *Is the map $H\colon C_0 \times [0,1] \to C_0$ defined by $H(\bar{v}, t) = G_t(\bar{v})$ a completely continuous map? Is it continuous with respect to the product topology in $C_0 \times [0,1]$?*

Problem. *An interesting problem is to understand how these techniques translate to the theory of Anosov flows. Does there exist a Banach space and a map on this space such that an analogous of Theorem 3 holds? There are some examples of Anosov flows that are not product Anosov flows (see [10], [20]) so if we are able to obtain this map it should not admit fixed points.*

Proof of Theorem 16

Now we turn our attention to the proof of Theorem 16. For each $k \in \mathbb{Z}$ we consider projections $\pi_k^s\colon T_{x_k}\widetilde{M} \to E^s(x_k)$ and $\pi_k^u\colon T_{x_k}\widetilde{M} \to E^u(x_k)$ parallel to $E^u(x_k)$ and $E^s(x_k)$, respectively. Since M is compact we can choose $N \in \mathbb{N}$ such that for every $k \in \mathbb{Z}$ we have

$$|\pi_k^s(v)|_{x_k} \leq N|v|_{x_k} \quad \text{and} \quad |\pi_k^u(v)|_{x_k} \leq N|v|_{x_k}.$$

For each $k \in \mathbb{Z}$ consider the map $A_k\colon T_{x_k}\widetilde{M} \to T_{x_{k+1}}\widetilde{M}$ defined by

$$A_k(v) = I_{k+1} \circ Df(x_k)(v).$$

Since $A_k(E^s(x_k)) = E^s(x_{k+1})$ for every $k \in \mathbb{Z}$ we can compose these maps to obtain

$$A_{k-1} \circ \cdots \circ A_n(E^s(x_n)) = E^s(x_k), \quad n < k.$$

Note also that $A_k^{-1}(E^u(x_{k+1})) = E^u(x_k)$ so we analogously have

$$A_k^{-1} \circ \cdots \circ A_n^{-1}(E^u(x_{n+1})) = E^u(x_k), \quad n \geq k.$$

Thus we can define a map $\mathcal{G}\colon C_0 \to C_0$ as follows: for each $\bar{v} = (v_k)_{k\in\mathbb{Z}} \in C_0$ the sequence $\mathcal{G}(\bar{v}) = (\mathcal{G}(\bar{v})_k)_{k\in\mathbb{Z}}$ is defined by

$$\mathcal{G}(\bar{v})_k = \pi_k^s(v_k) + \sum_{n=-\infty}^{k-1} A_{k-1}\circ\cdots\circ A_n\pi_n^s(v_n) - \sum_{n=k}^{+\infty} A_k^{-1}\circ\cdots\circ A_n^{-1}\circ\pi_{n+1}^u(v_{n+1}).$$

We will prove that $\mathcal{G}$ is the inverse of the map $Id - T$. First note that for every $v \in E^s(x_k)$, $w \in E^u(x_{k+1})$ and $k \in \mathbb{Z}$ we have

$$|A_k(v)|_{x_{k+1}} \leq \lambda|v|_{x_k} \quad \text{and} \quad |A_k^{-1}(w)|_{x_k} \leq \lambda|w|_{x_{k+1}}.$$

Hence for every $v \in E^s(x_n)$ and $n < k$ we have

$$|A_{k-1}\circ\cdots\circ A_n(v)|_{x_k} \leq \lambda^{k-n}|v|_{x_n}.$$

Also for every $w \in E^u(x_{n+1})$ and $n \geq k$ we have

$$|A_k^{-1}\circ\cdots\circ A_n^{-1}(w)|_{x_k} \leq \lambda^{n-k+1}|w|_{x_{n+1}}.$$

Therefore

$$\begin{aligned}
\left|\sum_{n=-\infty}^{k-1} A_{k-1}\circ\cdots\circ A_n\pi_n^s(v_n)\right|_{x_k} &\leq \sum_{n=-\infty}^{k-1}\lambda^{k-n}|\pi_n^s(v_n)|_{x_n}\\
&\leq N\sum_{n=-\infty}^{k-1}\lambda^{k-n}|v_n|_{x_n}\\
&\leq N||\bar{v}||\sum_{n=-\infty}^{k-1}\lambda^{k-n}\\
&\leq N||\bar{v}||\frac{\lambda}{1-\lambda}
\end{aligned}$$

and

$$\begin{aligned}
\left|\sum_{n=k}^{+\infty} A_k^{-1}\circ\cdots\circ A_n^{-1}\circ\pi_{n+1}^u(v_{n+1})\right|_{x_k} &\leq \sum_{n=k}^{+\infty}\lambda^{n-k+1}|\pi_{n+1}^u(v_{n+1})|_{x_{n+1}}\\
&\leq N\sum_{n=k}^{+\infty}\lambda^{n-k+1}|v_{n+1}|_{x_{n+1}}\\
&\leq N||\bar{v}||\sum_{n=k}^{+\infty}\lambda^{n-k+1}\\
&\leq N||\bar{v}||\frac{\lambda}{1-\lambda}.
\end{aligned}$$

Thus for every $k \in \mathbb{Z}$ we have

$$|\mathcal{G}(\bar{v})_k|_{x_k} \leq N||\bar{v}|| + N||\bar{v}||\frac{\lambda}{1-\lambda} + N||\bar{v}||\frac{\lambda}{1-\lambda} = N\left(\frac{1+\lambda}{1-\lambda}\right)||\bar{v}||.$$

This proves that $\mathcal{G}(\bar{v})_k \in T_{x_k}\widetilde{M}$ for every $k \in \mathbb{Z}$. We prove more:

Proposition 6. *If $\bar{v} \in C_0$ then $\mathcal{G}(\bar{v}) \in C_0$.*

First we prove an auxiliary lemma:

Lemma 13. *If $\bar{v} = (v_k)_{k\in\mathbb{Z}} \in C_0$ then $\sum_{n=1}^{k-1} \lambda^{k-n}|v_n|_{x_n} \to 0$ when $k \to \infty$.*

Proof. For each $\varepsilon > 0$ choose $K \in \mathbb{N}$ such that $|v_n|_{x_n} < \frac{\varepsilon(1-\lambda)}{2}$ for every $n \geq K$. For $k > K$ we write

$$\sum_{n=1}^{k-1} \lambda^{k-n}|v_n|_{x_n} = \sum_{n=1}^{K} \lambda^{k-n}|v_n|_{x_n} + \sum_{n=K+1}^{k-1} \lambda^{k-n}|v_n|_{x_n}.$$

Note that

$$\sum_{n=1}^{K} \lambda^{k-n}|v_n|_{x_n} = \lambda^k \left(\sum_{n=1}^{K} \lambda^{-n}|v_n|_{x_n}\right) \to 0, \quad k \to \infty.$$

So we can choose $k \geq K$ such that

$$\sum_{n=1}^{K} \lambda^{k-n}|v_n|_{x_n} < \frac{\varepsilon}{2}.$$

Moreover,

$$\begin{aligned}
\sum_{n=K+1}^{k-1} \lambda^{k-n}|v_n|_{x_n} &\leq \frac{\varepsilon(1-\lambda)}{2}\left(\sum_{n=K+1}^{k-1} \lambda^{k-n}\right) \\
&\leq \frac{\varepsilon(1-\lambda)}{2}\left(\frac{1}{1-\lambda}\right) \\
&= \frac{\varepsilon}{2}.
\end{aligned}$$

Thus for each $\varepsilon > 0$ there is $k \in \mathbb{N}$ such that

$$\sum_{i=1}^{n-1} \lambda^{n-i}|v_i|_{x_i} < \varepsilon, \;\; n \geq k.$$

This finishes the proof. □

Proof of Proposition 6: Let $\bar{v} = (v_k)_{k\in\mathbb{Z}} \in C_0$. To prove that $\mathcal{G}(\bar{v}) \in C_0$ we need to show that $|\mathcal{G}(\bar{v})_k|_{x_k} \to 0$ when $|k| \to \infty$. To prove this we consider separately the three terms in the expression of $\mathcal{G}(\bar{v})_k$. The first one satisfies

$$|\pi_k^s(v_k)|_{x_k} \leq N|v_k|_{x_k} \to 0, \quad |k| \to \infty.$$

For the second term it is enough to prove that

$$\sum_{n=-\infty}^{k-1} \lambda^{k-n}|v_n|_{x_n} \to 0, \quad |k| \to \infty.$$

For $k > 1$ we write

$$\sum_{n=-\infty}^{k-1} \lambda^{k-n}|v_n|_{x_n} = \sum_{n=-\infty}^{0} \lambda^{k-n}|v_n|_{x_n} + \sum_{n=1}^{k-1} \lambda^{k-n}|v_n|_{x_n}.$$

The previous lemma shows that the second sum goes to zero when $k \to \infty$. For the first sum we have

$$\begin{aligned}\sum_{n=-\infty}^{0} \lambda^{k-n}|v_n|_{x_n} &= \lambda^k \left(\sum_{n=-\infty}^{0} \lambda^{-n}|v_n|_{x_n}\right) \\ &\leq \lambda^k||\bar{v}|| \left(\sum_{n=-\infty}^{0} \lambda^{-n}\right) \\ &= \lambda^k||\bar{v}|| \left(\frac{1}{1-\lambda}\right) \to 0, \quad k \to \infty.\end{aligned}$$

For $k < 0$ we argument as follows. For each $\varepsilon > 0$ we choose $K \in \mathbb{N}$ such that

$$|v_n|_{x_n} < \frac{\varepsilon(1-\lambda)}{\lambda}, \quad n \leq -K.$$

Thus for $k \leq -K$ we have

$$\begin{aligned}\sum_{n=-\infty}^{k-1} \lambda^{k-n}|v_n|_{x_n} &\leq \frac{\varepsilon(1-\lambda)}{\lambda} \left(\sum_{n=-\infty}^{k-1} \lambda^{k-n}\right) \\ &= \frac{\varepsilon(1-\lambda)}{\lambda} \left(\frac{\lambda}{1-\lambda}\right) \\ &= \varepsilon.\end{aligned}$$

This proves the desired convergence. For the third term in $\mathcal{G}(\bar{v})_k$ we can use the same arguments so we leave the details to the reader. □

Lemma 14. *For each $\bar{v} \in C_0$ and each $k \in \mathbb{Z}$ the following holds:*

$$A_k(\mathcal{G}(\bar{v})_k) = \mathcal{G}(\bar{v})_{k+1} - v_{k+1}.$$

Proof. For each $k \in \mathbb{Z}$ we have

$$\begin{aligned}A_k(\mathcal{G}(\bar{v})_k) = A_k \circ \pi^s_k(v_k) \quad &+ \sum_{n=-\infty}^{k-1} A_k \circ A_{k-1} \circ \cdots \circ A_n \pi^s_n(v_n) - \pi^u_{k+1}(v_{k+1}) \\ &- \sum_{n=k+1}^{+\infty} A^{-1}_{k+1} \circ \cdots \circ \pi^u_{n+1}(v_{n+1}).\end{aligned}$$

To obtain the desired equality just put $-\pi^u_{k+1}(v_{k+1}) = \pi^s_{k+1}(v_{k+1}) - v_{k+1}$ in the last one. □

Proof of Theorem 16: We first prove that $Id - T$ is surjective. Indeed, $\mathcal{G}$ is a right inverse for $Id - T$. For every $w \in C_0$ and $k \in \mathbb{Z}$ we have:

$$(Id - T)(\mathcal{G}(w))_k = \mathcal{G}(w)_k - A_{k-1}(\mathcal{G}(w)_{k-1}) = w_k,$$

where the last equality is ensured by Lemma 14. To prove that $Id - T$ is injective let $v \in C_0$ be such that $(Id - T)(v) = 0$, that is, $T(v) = v$. Then $A_{k-1}(v_{k-1}) = v_k$ for every $k \in \mathbb{Z}$. By induction we obtain

$$v_k = A_{k-1} \circ \cdots \circ A_0(v_0), \quad k > 0$$

and

$$v_k = A_k^{-1} \circ \cdots \circ A_{-1}^{-1}(v_0), \quad k < 0.$$

Now note that

$$|v_k|_{x_k} \to 0, \quad |k| \to \infty,$$

and that this implies

$$\pi_0^s(v_0) = \pi_0^u(v_0) = 0,$$

otherwise v_k would converge to ∞. Therefore $v_0 = 0$, which implies $v_k = 0$ for every $k \in \mathbb{Z}$ and we are done. □

Remark 11. *The map $\mathcal{G}$ is a linear isomorphism in C_0 that is the inverse of $Id - T$ and has norm*

$$||\mathcal{G}|| = \sup_{||\bar{v}||=1} ||\mathcal{G}(\bar{v})|| = \sup_{||\bar{v}||=1} \sup_{k\in\mathbb{Z}} |\mathcal{G}(\bar{v})_k|_{x_k} \leq N\frac{1+\lambda}{1-\lambda}.$$

Remark 12. *If f is a partially hyperbolic diffeomorphism then we can consider the same operator $\mathcal{G}$ as defined above. The difference is that Lemma 14 does not hold as it is written. Indeed, the following holds: for each $\bar{v} \in C_0$ and each $k \in \mathbb{Z}$ we have*

$$A_k(\mathcal{G}(\bar{v})_k) = \mathcal{G}(\bar{v})_{k+1} - v_{k+1} + \pi_{k+1}^c(v_{k+1}),$$

where π_k^c is the projection in the central direction $E^c(x_k)$ parallel to $E^s(x_k) \oplus E^u(x_k)$. In this case we have

$$(Id - T)(\mathcal{G}(w))_k = \mathcal{G}(w)_k - A_{k-1}(\mathcal{G}(w)_{k-1}) = w_k - \pi_k^c(w_k)$$

and $\mathcal{G}$ is not the inverse of $Id - T$ anymore.

Chapter 5

Non-Anosov scenario

On this chapter we discuss the presence of the two-sided limit shadowing property for diffeomorphisms outside the class of Anosov diffeomorphisms. The first class we discuss is the class of Axiom A diffeomorphisms.

Definition 17. *We say that a diffeomorphism f on a compact and connected manifold M is* Axiom A *when the non-wandering set of f is hyperbolic*[1] *and the set of periodic points*[2] *of f is dense in the non-wandering set.*

It is clear that Anosov diffeomorphisms are Axiom A diffeomorphisms. These systems do not have a hyperbolic structure on the whole manifold M as the Anosov diffeomorphisms but they still have some shadowing properties due to its relation to the stability theory (see [37], [43], [48]). It is obvious that transitive Axiom A diffeomorphisms are Anosov diffeomorphisms, so a consequence of Theorem 1 is the following:

Corollary 9. *Axiom A diffeomorphisms that are not Anosov do not admit the two-sided limit shadowing property.*

There exist diffeomorphisms that admit the two-sided limit shadowing property outside the class of Anosov diffeomorphisms. In [33] and [39] one can find examples of diffeomorphisms that are not Anosov but are topologically conjugated[3] to a transitive Anosov diffeomorphism. These systems have the two-sided limit shadowing property by the following lemma:

Lemma 15. *If f and h are homeomorphisms on a compact metric space X then f has the two-sided limit shadowing property if and only if so does $g = h \circ f \circ h^{-1}$.*

[1] A compact f-invariant set Λ is hyperbolic when properties (1) and (2) on the definition of Anosov diffeomorphisms hold on points of Λ.

[2] We recall that a point $p \in M$ is *periodic* for f if there exists $n \in \mathbb{N}$ such that $f^n(p) = p$. The set of all periodic points of f is usually denoted by $Per(f)$.

[3] Two maps f and g on a metric space X are *topologically conjugated* if there exists a homeomorphism h on X satisfying $g = h \circ f \circ h^{-1}$.

Proof. It is sufficient to prove one of the implications. We suppose f has the two-sided limit shadowing and prove g also has it. Let $(x_k)_{k\in\mathbb{Z}}$ be a two-sided limit pseudo-orbit for g. We claim that $(h^{-1}(x_k))_{k\in\mathbb{Z}}$ is a two-sided limit pseudo-orbit for f. To see this fix any $\varepsilon > 0$ and choose $\delta > 0$ such that $d(x, y) < \delta$ implies $d(h^{-1}(x), h^{-1}(y)) < \varepsilon$. Choose $N \in \mathbb{N}$ such that if $|k| > N$ then $d(g(x_k), x_{k+1}) < \delta$. Thus if $|k| > N$ we have

$$d(f(h^{-1}(x_k)), h^{-1}(x_{k+1})) = d(h^{-1}(g(x_k)), h^{-1}(x_{k+1})) < \varepsilon.$$

As for each $\varepsilon > 0$ there exists $N \in \mathbb{N}$ satisfying above inequality we have

$$d(f(h^{-1}(x_k)), h^{-1}(x_{k+1})) \to 0, \quad |k| \to \infty.$$

This proves that $(h^{-1}(x_k))_{k\in\mathbb{Z}}$ is a two-sided limit pseudo-orbit for f. Since f has the two-sided limit shadowing property there exists $z \in M$ satisfying

$$d(f^k(z), h^{-1}(x_k)) \to 0, \quad |k| \to \infty.$$

Now we prove that $h(z)$ two-sided limit-shadows $(x_k)_{k\in\mathbb{Z}}$. Fix any $\alpha > 0$ and choose $\beta > 0$ such that $d(x, y) < \beta$ implies $d(h(x), h(y)) < \alpha$. Choose $K \in \mathbb{N}$ such that if $|k| > K$ then $d(f^k(z), h^{-1}(x_k)) < \beta$. Thus if $|k| > K$ then

$$d(g^k(h(z)), x_k) = d(h(f^k(z)), h(h^{-1}(x_k))) < \alpha.$$

As for each $\alpha > 0$ there exists $K \in \mathbb{N}$ satisfying above inequality we have

$$d(g^k(h(z)), x_k) \to 0, \quad |k| \to \infty.$$

Since $(x_k)_{k\in\mathbb{Z}}$ was chosen arbitrarily it follows that g has the two-sided limit shadowing property. □

This proves that the two-sided limit shadowing property is invariant under topological conjugacies and is present in any system that is topologically conjugated to a transitive Anosov diffeomorphism. Furthermore, we have the following question.

Question 7. *Is it true that any diffeomorphism with the two-sided limit shadowing property must be topologically conjugated to an Anosov diffeomorphism?*

Remark 13. *A similar problem was posed in Conjecture 1 for the Torus $\mathbb{T}^n$, where we think expansiveness is a necessary condition for the presence of the two-sided limit shadowing property. Indeed, the main theorem in [31] says that any expansive homeomorphism on the Torus with the shadowing property is topologically conjugated to an Anosov diffeomorphism. So a positive answer for above question would give a positive answer to Conjecture 1.*

We denote by $\mathrm{Diff}^1(M)$ the set of all C^1 diffeomorphisms on a compact manifold M endowed with the usual C^1 topology. The following holds.

Theorem 17. *There exists a residual subset of $\mathrm{Diff}^1(M)$ such that diffeomorphisms in this residual that are not Anosov cannot be topologically conjugated to a transitive Anosov diffeomorphism.*

Proof. The main theorem in [8] proves the existence of a residual subset of $\mathrm{Diff}^1(M)$ such that diffeomorphisms in this residual that are not Axiom A cannot be expansive. If a diffeomorphism that is topologically conjugated to a transitive Anosov diffeomorphism belongs to this residual then it would be Axiom A, since it is expansive. Moreover, it would have the two-sided limit shadowing property (by Theorem 14 and Lemma 15). So Corollary 9 shows it is Anosov, which contradicts the hypothesis. □

Remark 14. *If Question 7 ends to be true then Theorem 17 would imply Theorem 4.*

Since we do not have an answer to above question we prove Theorem 4 below. A central property in the proof of this theorem is the *star* condition, that we define now properly.

Definition 18. *A periodic point of* $f \in \mathrm{Diff}^1(M)$ *is called* hyperbolic *if its orbit is a hyperbolic set. We say* f *is a* star diffeomorphism *if there exists a* C^1 *neighborhood* $\mathcal{U}$ *of* f *such that every periodic point of every* $g \in \mathcal{U}$ *is hyperbolic.*

In other words, the star condition means that the hyperbolicity of all periodic points persist under small perturbations. In a remarkable paper [29] S. Hayashi proved that the class of star diffeomorphisms is contained in the class of Axiom A diffeomorphisms. Our proof is based on this fact and we will be interested in proving that the two-sided limit shadowing property implies the star condition in some sense. If this is the case then Corollary 9 assures they must be Anosov.

We denote by $\mathcal{TLS}$ the set of all diffeomorphisms with the two-sided limit shadowing property. Before proving Theorem4 we prove the following:

Theorem 18 ([15])**.** *The* C^1*-interior of* $\mathcal{TLS}$ *is equal to the set of transitive Anosov diffeomorphisms.*

This shows that there are no open sets in $\mathcal{TLS}$ outside the set of transitive Anosov diffeomorphisms, or equivalently, that systems without the two-sided limit shadowing property are dense in the complement of the set of transitive Anosov diffeomorphisms. To prove this theorem we make some definitions and state some results.

Definition 19. *The* index *of a hyperbolic periodic point* p *is the dimension of the stable space of* p *and is denoted by* $ind(p)$.

We state a well known lemma in dynamics that bifurcates a non-hyperbolic periodic point into two distinct hyperbolic periodic points with different indices. A proof can be found in [54] Lemma 2.4.

Lemma 16. *If* $f \in \mathrm{Diff}^1(M)$ *and* p *is a non-hyperbolic periodic point of* f *then for every* C^1 *neighborhood* $\mathcal{U}$ *of* f *there are* $g \in \mathcal{U}$ *and* $p_1, p_2 \in Per(g)$ *such that* $ind(p_1) \neq ind(p_2)$.

Definition 20. *We say that a diffeomorphism* f *is Kupka-Smale if all periodic points of* f *are hyperbolic and for all pair of periodic points* p *and* q *of* f *the manifolds* $W^s(p)$ *and* $W^u(q)$ *are transversal.*

We denote by $\mathcal{KS}$ the set of all Kupka-Smale diffeomorphisms. It is well known that $\mathcal{KS}$ is a residual subset of $\mathrm{Diff}^1(M)$.

Definition 21. *We say that two hyperbolic periodic points p and q are* homoclinicaly related *if the stable manifold of the orbit*[4] *of p and the unstable manifold of the orbit of q intersect transversally and also the unstable manifold of the orbit of p and the stable manifold of the orbit of q intersect transversally.*

Remark 15. *It is easy to see that two hyperbolic periodic points homoclinically related have the same index.*

The following is an important lemma.

Lemma 17. *If $f \in \mathcal{TLS} \cap \mathcal{KS}$ then all periodic points of f have the same index.*

Proof. For any periodic points p and q of f, the two-sided limit shadowing property assures the sets $W^s(p) \cap W^u(q)$ and $W^u(p) \cap W^s(q)$ are non-empty. Since $f \in \mathcal{KS}$ these intersections are transversal. Hence p and q are homoclinicaly related and have the same index (see Remark 15). □

Proof of Theorem 18: Let $f \in int(\mathcal{TLS})$ and $\mathcal{U}$ be a C^1 neighborhood of f contained in $\mathcal{TLS}$. We claim that f is a star diffeomorphism. Suppose by contradiction that f is not star. Then there exist $g \in \mathcal{U}$ and p a non-hyperbolic periodic point of g. Lemma 16 assures the existence of $h \in \mathcal{U}$ and two distinct hyperbolic periodic points p, q of h with different indices. As these points are hyperbolic and $\mathcal{KS}$ is dense in $\mathrm{Diff}^1(M)$ we can perturb h to $\tilde{h} \in \mathcal{U} \cap \mathcal{KS}$ that has two distinct hyperbolic periodic points $p_{\tilde{h}}, q_{\tilde{h}}$ with different indices. This contradicts Lemma 17 and proves the claim. Corollary 9 finishes the proof. □

To prove Theorem 4 we first introduce some generic machinery.

Lemma 18. *There exists an open and dense set $\mathcal{P}$ of $\mathrm{Diff}^1(M)$ such that every $f \in \mathcal{P}$ satisfies: if for any C^1 neighborhood $\mathcal{U}$ of f some $g \in \mathcal{U}$ has two distinct hyperbolic periodic points with different indices then f has two distinct hyperbolic periodic points with different indices.*

Proof. Let $\mathcal{A}$ be the set of all diffeomorphisms admitting two distinct hyperbolic periodic points with different indices. The hyperbolicity of these periodic points imply that $\mathcal{A}$ is open in $\mathrm{Diff}^1(M)$. If $\mathcal{B} = \mathrm{Diff}^1(M) \setminus \overline{\mathcal{A}}$ then $\mathcal{B}$ is also open in $\mathrm{Diff}^1(M)$. Thus $\mathcal{P} = \mathcal{A} \cup \mathcal{B}$ is open and dense in $\mathrm{Diff}^1(M)$.

For $f \in \mathcal{P}$ let $(f_n)_{n\in\mathbb{N}}$ be a sequence of diffeomorphisms converging to f in the C^1 topology such that each f_n has two distinct hyperbolic periodic points with different indices. Thus $f_n \in \mathcal{A}$ for all $n \in \mathbb{N}$ and $f \in \mathcal{P}$ imply that f cannot belong to $\mathcal{B}$ and must belong to $\mathcal{A}$. Hence f has two distinct hyperbolic periodic points with different indices. □

[4]The stable set of the orbit of a periodic point is, by definition, the union of the stable sets of each point of his orbit. The same holds for the unstable set of the orbit of p.

Proof of Theorem 4: It is enought to prove that diffeomorphisms in $\mathcal{TLS} \cap \mathcal{KS} \cap \mathcal{P}$ are star diffeomorphisms. If there exists $f \in \mathcal{TLS} \cap \mathcal{KS} \cap \mathcal{P}$ that is approximated in the C^1-topology by diffeomorphisms admitting a non-hyperbolic periodic point then Lemma 16 assures it is approximated by diffeomorphisms admitting periodic points with different indices. Since $f \in \mathcal{P}$ it follows that f admits two periodic points with different indices, which is a contradiction with Corollary 17. Hence f is a star diffeomorphism. □

Chapter 6

Appendix

On this appendix we exhibit a joint work with A. Arbieto, W. Cordeiro and D. Obata, named "On bi-Lyapunov stable homoclinic classes" that is published in the Bulletin of the Brazilian Mathematical Society [9].

Introduction and Statement of the Results

One of the most important theorem on hyperbolic dynamics is the *spectral decomposition* theorem, proved by Smale [56]. It says that the non-wandering set of an Axiom A diffeomorphism is a disjoint union of transitive invariant compact sets. Actually, the proof show that these sets are *homoclinic classes* of a periodic point inside them. Since then, the study of homoclinc classes for more general dynamics attracts a lot of attention by many researches. Indeed, to decide if such classes are hyperbolic under some dynamical hypothesis is an important problem. One of the most famous is Palis' conjecture, in [47], which says that a diffeomorphism far away from homoclinic tangencies and heterodimensional cycles can be approximated by an Axiom A diffeomorphism. In particular, if the diffeomorphism is generic, the spectral decomposition theorem, cited above, says that its homoclinic classes are hyperbolic. We remark that this conjecture was proved in [52] for surface diffeomorphisms.

The spectral decomposition theorem also says that the classes are *isolated*. So, many efforts were done to understand isolated homoclinic classes for generic diffeomorphisms and there are many satisfactory results about them. Actually, this leads to the study of *tame* diffeomorphisms, like in [2]. Such diffeomorphisms has the property that they have only a finite number of chain recurrence classes, and Abdenur shows that Palis' conjecture is true in the set of tame diffeomorphisms, see [2].

It is natural to study the complementary class, the so called *wild* diffeomorphisms. In particular, to study non-isolated homoclinic classes. This is an important problem, since many rich dynamics can be generated from such classes with lack of hyperbolicity, see for example [13].

However, one can study an intermediary case between this scenario and the tame scenario, looking for the dynamics of bi-Lyapunov stable homoclinic classes, as Potrie did in [50]. Actually, Potrie showed that such classes always have a dominated splitting. The first result in this note is to notice that under the hypothesis of homogeneity, weak eigenvalues can be ruled out using Potrie's argument.

Theorem A. *Let f be a C^1 generic diffeomorphism. If Λ is a bi-Lyapunov stable homogeneous homoclinic class then there exists $\delta > 0$ such that any periodic point contained in the class has no δ-weak eigenvalue.*

As an application, we will prove that such homoclinic classes are in fact hyperbolic provided that they have some other dynamical property. These properties are well known in the theory of dynamical systems. Indeed, shadowing and limit shadowing are important tools in stability theory [49] and the specification property was used in the development of the thermodynamical formalism, see [30], and also is present in the study of invariant measures, see [55].

Theorem B. *Let f be a C^1 generic diffeomorphism. If Λ is a bi-Lyapunov stable homoclinic class which either has the shadowing property, or the specification property, or the limit shadowing property, then it is hyperbolic. Furthermore, Λ is the whole manifold and f is Anosov.*

In fact, these results are somewhat connected with two conjectures about homoclinic classes. The first one was proposed in [5].

Conjecture 1. *Let f be a C^1 generic diffeomorphism and Λ a homoclinic class which has the shadowing property. Then Λ is hyperbolic.*

Actually, in [5] they proved this conjecture for isolated homoclinic classes. The second conjecture was proposed in [3]:

Conjecture 2. *Let f be a C^1-generic diffeomorphism. If $\Omega(f)$ has nonempty interior then f is transitive, in particular the $\Omega(f) = M$.*

As is remarked in [3], if $\Omega(f)$ has non-empty interior for some generic f then there exists a homoclinic class with non-empty interior. So, it is sufficient to prove that for a generic diffeomorphism if a homoclinic class has non-empty interior then the diffeomorphism is transitive. In [3] the conjecture is proved for isolated homoclinic classes, as well for strongly partially hyperbolic homoclinic classes. In [51], the authors prove the conjecture in any dimension for classes which their finest dominated splitting have one-dimensional extreme sub bundles. Potrie in [50] solved the conjecture in dimension 3, studying bi-Lyapunov stable homoclinic classes.

A corollary of Theorem B is the following.

Corollary C. *If f is a C^1 generic diffeomorphism with a homoclinic class with nonempty interior satisfying the shadowing property then it is Anosov.*

The paper is organized as follows. In section 2, we give precise definitions, including the ones used in the introduction. We also collect some results that will be used. In section 3, we prove theorem A. In section 4, we prove theorem B using two propositions, whose proofs will be postponed to sections 5 and 6.

Definitions and some Tools

In this section we define precisely the objects that appear in the statements of the theorems in the introduction and collect some results which will be needed later.

Domination and Hyperbolicity

Let K be a compact and invariant set. A linear subbundle E of the tangent bundle $T_K M$ is *uniformly contracted* by f if it is Df-invariant and there exists $N \geq 1$ such that for any $x \in K$ and any unit vector $v \in E_x$ we have

$$||Df^N(x)v|| < \frac{1}{2}.$$

If E is uniformly contracted for f^{-1} we say that it is *uniformly expanded.*

The set K is *hyperbolic* if $T_K M = E \oplus F$, where E is uniformly contracted and F is uniformly expanded. We call E the stable bundle and F the unstable bundle. If the whole manifold M is hyperbolic we say that f is *Anosov.*

We say that the tangent bundle over K has a *dominated splitting* if $T_K M = E \oplus F$, where E and F are Df-invariant subbundles and there exists $N \geq 1$ such that for all $x \in K$ and any unit vectors $u \in E$ and $v \in F$ we have

$$||Df^N(x)u|| < \frac{1}{2}||Df^N(x)v||.$$

A periodic point p is called hyperbolic if its orbit is a hyperbolic set. We define the index of a hyperbolic periodic point as the dimension of the stable bundle.

Given $0 < \delta < 1$, we say that p has a δ-weak eigenvalue, if $Df^{\pi(p)}(p)$ has an eigenvalue μ such that $(1-\delta)^{\pi(p)} < |\mu| < (1+\delta)^{\pi(p)}$.

Finally, a diffeomorphism is *hyperbolic* if its chain recurrent set is hyperbolic.

Recurrence

We recall that a sequence $\{x_n\}_{n=a}^{b}$ is an ε-*pseudo orbit* of f if for every j we have $d(f(x_j), x_{j+1}) < \delta$, where $-\infty \le a \le j < b \le +\infty$.

The *chain recurrent* set of a diffeomorphism f, denoted by $R(f)$, is the set of points $x \in M$ such that for each $\varepsilon > 0$ exists a finite ε-pseudo orbit such that $x_0 = x$ and $x_k = x$. The chain recurrent set contains all the periodic points, recurrent points, and non-wandering points.

It is known that $R(f)$ can be decomposed into a disjoint union of compact, invariant and "non-decomposable" sets. More precisely, we define a relation $\leftrightsquigarrow$ on $R(f)$ by $x \leftrightsquigarrow y$ if for every $\varepsilon > 0$ there exists an ε-pseudo orbit $(x_n)_{n=0}^{n=k}$ such that $x_0 = x$ and $x_k = y$ and another ε-pseudo orbit $(y_n)_{n\in\mathbb{N}}$ such that $y_0 = y$ and $y_n = x$. Clearly $\leftrightsquigarrow$ is an equivalence relation on $R(f)$. The equivalence classes are called the *chain recurrent classes* of f.

The *homoclinic class* of a hyperbolic periodic point p is the closure of the transverse intersection of the stable and unstable manifolds of p. It can also be defined as the closure of the set of periodic points that are homoclinically related to p. We say that two hyperbolic periodic orbits O and O' are homoclinically related if $W^s(O) \pitchfork W^u(O') \neq \emptyset$ and $W^u(O) \pitchfork W^s(O') \neq \emptyset$. Homoclinic classes are *transitive* sets, i.e. there exist a point in the class whose orbit is dense. We say that a homoclinic class is *homogeneous* if all the periodic points of this class have the same index.

An invariant and compact subset $\Lambda \subset M$ is *Lyapunov stable* for f if for every neighborhood U of Λ there is another neighborhood V of Λ such that $f^n(V) \subset U$ for any $n \ge 1$. We say that Λ is *bi-Lyapunov stable* if it is Lyapunov stable for f and for f^{-1}.

Shadowing, specification and limit shadowing properties

In this section, we give the precise definitions of the dynamical properties that were used in the statement of the main theorems.

We start with the shadowing property.

Definition 1. *A compact invariant set Λ has the shadowing property, or simply has shadowing, if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every δ-pseudo orbit $\{x_n\}_{n\in\mathbb{Z}} \subset \Lambda$ there exists $x \in M$ that ε-shadows $\{x_n\}_{n\in\mathbb{Z}}$, that is, $d(f^n(x), x_n) < \varepsilon$ $\forall n \in \mathbb{Z}$. We say that a diffeomorphism f has the shadowing property or is shadowable if M has the shadowing property.*

Now, we turn to the specification property, see [28]. First, we recall what a specification is.

Definition 2. *A specification $S = (\tau, P)$ consists of a finite collection $\tau = \{I_1, \dots, I_m\}$ of finite intervals $I_i = [a_i, b_i] \subset \mathbb{Z}$ and a map $P\colon \bigcup_{i=1}^{m} I_i \to M$ such that for each $t_1, t_2 \in I \in \tau$ we have $f^{t_2 - t_1}(P(t_1)) = P(t_2)$. The specification S is said to be L-spaced if $a_{i+1} \ge b_i + L$ for all $i \in \{1, \dots, m\}$. Moreover, a specification S is ε-shadowed by $x \in M$ if $d(f^n(x), P(n)) < \varepsilon$ for all $n \in \bigcup_{i=1}^{m} I_i$.*

Now, we define the specification property.

Definition 3. *If Λ is a compact metric space we say that $f|_\Lambda$ has the specification property if for every $\varepsilon > 0$ there exists $M \in \mathbb{N}$ such that every M-spaced specification such that $P(a_i), P(b_i) \in \Lambda$ is ε-shadowed by $x \in \Lambda$. If $\Lambda = M$ we just say that f has the specification property.*

Finally, we turn to the limit shadowing property.

Definition 4. *We say that a diffeomorphism f has the limit shadowing property if for any sequence $(x_i)_{i\in\mathbb{Z}} \subset M$ such that $d(f(x_i), x_{i+1}) \to 0$ as $|i| \to \infty$ there exists a point $y \in M$ such that $d(f^i(y), x_i) \to 0$ as $|i| \to \infty$. The sequence $(x_i)_{i\in\mathbb{Z}} \subset M$ is called a limit pseudo-orbit.*

Gourmelon-Franks' lemma

In this section, we state a Frank's-type lemma due to Gourmelon, extending [22]. Actually, we use the approach of Potrie in [50].

A 4-tuple $\mathcal{A} = (\Sigma, f, E, A)$ is a *large period linear cocycle* of dimension d bounded by K over an infinite set Σ, or simply a *cocycle*, if

- Σ is a set and $f\colon \Sigma \to \Sigma$ is a one-to-one map such that all points in Σ are periodic and such that given $n > 0$ there are only finitely many period less than n;
- $\pi :: E \to \Sigma$ is a linear bundle of dimension d over Σ, whose fibers are endowed with a euclidian metric $||.||$. The fiber over the point $x \in \Sigma$ will be denoted by E_x.
- $A :: x \in \Sigma \to A_x \in GL(E_x, E_{f(x)})$ is such that $||A_x|| \le K$ and $||A_x^{-1}|| \le K$.

For every $x \in \Sigma$ and $1 \le j \le d$, denote by $\pi(x)$ its period and define the j-th Lyapunov exponent of $\mathcal{A}$ at x by

$$\sigma^j(x, \mathcal{A})\colon = \frac{\log|\lambda_j|}{\pi(x)},$$

where $\lambda_1, ...\lambda_d$ are the eigenvalues of $A_x^{\pi(x)}$ in increasing order of modulus. We say that the cocycle $\mathcal{A}_x$ has a *i-strong stable manifold* if

$$\sigma^i(x, \mathcal{A}_x) < \min\{0, \sigma^{i+1}(x, \mathcal{A}_x)\}.$$

Define Γ_x as the set of cocycles over the orbit of x. We endow it with the following distance:

$$d(\mathcal{A}_x, \mathcal{B}_x) = \sup_{0\le i\le\pi(x), v\in E-0} \left\{ \frac{||(A_{f^i(x)} - B_{f^i(x)})v||}{||v||}, \frac{||(A^{-1}_{f^i(x)} - B^{-1}_{f^i(x)})v||}{||v||} \right\}$$

We also define $\Gamma_{x,i} = \{\mathcal{A}_x \in \Gamma_x; \mathcal{A}_x\ has\ a\ i-strong\ stable\ manifold\}$ and set Γ_Σ as the set of all cocyles over all of the orbits of Σ.

We say that $\mathcal{B}$ is a *perturbation* of $\mathcal{A}$ if for every $\varepsilon > 0$ the set of points in Σ such that $\mathcal{B}_x$ is not $\varepsilon-$close to the cocycle $\mathcal{A}_x$ is finite. Similarly, we say that $\mathcal{B}$ is a *path perturbation* of $\mathcal{A}$ if for every $\varepsilon > 0$ one has that the set of points $x \in \Sigma$ such that $\mathcal{B}_x$ is not a perturbation of $\mathcal{A}_x$ along a path of diameter $\leq \varepsilon$ is finite. That is, there is a path $\gamma\colon [0,1] \to \Gamma_\Sigma$ such that $\gamma(0) = \mathcal{A}$, $\gamma(1) = \mathcal{B}$, $\gamma_x\colon [0,1] \to \Gamma_x$ are continuous paths and given $\varepsilon > 0$ the set of x such that $\gamma_x([0,1])$ has diameter greater than ε is finite. Now, we have the notion of a path inside de the set $\Gamma_{x,i}$ and we can perturb the cocycles along a periodic orbit.

Let g be a perturbation of f such that the cocycles Df_x and Dg_x are both in $\Gamma_{x,i}$ and let U be a neighborhood of $Orb(x)$. We say that g preserves locally the i-strong stable manifold of $Orb(x)$ outside U if the set of points of the i-strong stable manifold of $Orb(x)$ outside U whose positive iterates do not leave U once they entered it are the same for f and for g.

We have the following theorem due to Gourmelon [26]:

Theorem 1. *Let f be a diffeomorphism and O be a periodic orbit of f such that $D_O f \in \Gamma_i$ and let $\gamma\colon [0,1] \to \Gamma_i$ be a path starting at $D_O f$. Then, given a neighborhood U of O, there is a perturbation g of f such that $D_O g = \gamma(1)$, g coincides with f outside U and preserves locally the i-strong stable manifold of f outside U. Moreover, given $\mathcal{V}$ a C^1 neighborhood of f, there exists $\varepsilon > 0$ such that if $diam(\gamma) < \varepsilon$ one can choose $g \in \mathcal{V}$*

Some generic results

In this section we will list some C^1 generic properties.

Let X be a topological space. We call $R \subset X$ a residual subset of X if R contains an enumerable intersection of open and dense subsets of X. We say that a property is a generic property of X if it holds on a residual subset of X. When X is the space of diffeomorphisms $\mathrm{Diff}^1(M)$ endowed with the C^1 topology, and R is a generic subset of it where some property holds, we say that this property holds C^1-generically. Every diffeomorphism $f \in R$ is called a C^1-generic diffeomorphism.

We state some C^1-generic properties that will be very important for proving the results presented here:

There exists a residual subset $R \subset \mathrm{Diff}^1(M)$ such that every $f \in R$ satisfies:

1. f is Kupka-Smale, i.e. every periodic point is hyperbolic and their invariant manifolds intersect transversally, see [45].
2. The chain recurrent class that contains a periodic point p is $H(p, f)$, see [12].
3. If $q \in H(p, f)$ then $H(p, f) = H(q, f)$, see [14].
4. $H(p, f) = \overline{W^s(p)} \bigcap \overline{W^u(p)}$, see [14].
5. $\overline{W^s(p)}$ is Lyapunov stable for f^{-1} and $\overline{W^u(p)}$ is Lyapunov stable for f, see [14].

6. The application $g \mapsto H(p,g)$ is continuous in f, see [1].

7. If $H(p,f)$ is Lyapunov stable for f then there exists $\mathcal{U}$ a C^1 neighborhood of f such that $H(p_g,g)$ is Lyapunov stable for every $g \in \mathcal{U}$, see [50].

8. There exists a C^1 neighborhood $\mathcal{U}$ and an interval of natural numbers $[\alpha,\beta]$ such that for every $g \in \mathcal{U}$, $H(p_g,g)$ has periodic points of every index in $[\alpha,\beta]$. Also, every periodic point in $H(p,g)$ has its index in that interval, see [4].

Non-existence of weak eigenvalues

In this section we prove theorem A following ideas from [50].

Proof of theorem A. Let f be a C^1 generic diffeomorphism and $H(p,f)$ be a bi-Lyapunov stable homogeneous homoclinic class with index α. Theorem 1.1 in [50] asserts that $H(p,f)$ has a dominated splitting $E \oplus F$ such that $\alpha\colon = dim(E) = ind(p)$. Generic properties (7) and (8) implies the existence of a C^1 neighborhood $\mathcal{U}$ of f such that for every $g \in \mathcal{U}$, $H(p_g,g)$ is bi-Lyapunov stable and every periodic point contained in $H(p_g,g)$ has the same index α.

By contradiction, let us assume that for each $\delta > 0$ there exists a periodic point q_δ homoclinically related to p with a δ-weak eigenvalue. Let $\varepsilon > 0$ be given by Gourmelon's theorem applied to the neighborhood $\mathcal{U}$. We will assume that the modulus of the weak eigenvalue of q_δ is between 1 and $1+\delta$ and we will use the Lyapunov stability for f. The other case is similar using the Lyapunov stability for f^{-1}. Furthermore, we can assume that the eigenvalues of q_δ are all real, using the next lemma.

Lemma 1 (Lemma 2.3 of [25])**.** *Let f be a C^1 generic diffeomorphisms. For any hyperbolic periodic point p of f , if for any $\delta > 0$, f has a periodic point q_δ homoclinically related to p with a δ-weak eigenvalue, then f has a periodic point p_1 homoclinically related to p with a δ-weak eigenvalue, whose eigenvalues are all real.*

Let v be the eigenvector related to the weak eigenvalue of q_δ. Using the splitting $T_{f^i(q_\delta)}M = [Df^i(q_\delta)v] \oplus [Df^i(q_\delta)v]^\perp$, we find a basis where:

$$Df(f^i(q_\delta)) = \begin{pmatrix} Df(f^i(q_\delta))|_{[Df^i(q_\delta)v]} & K_i^1(f) \\ 0 & K_i^2(f) \end{pmatrix}$$

Define $\gamma\colon [0,1] \to \Gamma_\alpha$ by

$$\gamma_i(t) = \begin{pmatrix} (1-t)Df(f^i(q_\delta))|_{[Df(q_\delta)^i v]} + t(\frac{1-\delta}{1+\delta})Df(f^i(q_\delta))|_{[Df^i(q_\delta)v]} & K_i^1(f) \\ 0 & K_i^2(f) \end{pmatrix}$$

We can choose δ small enough such that $diam(\gamma_i) < \varepsilon$. Note that,

$$\begin{aligned}\prod_{i=0}^{\pi(q_\delta)-1} \| \left(\frac{1-\delta}{1+\delta}\right) Df(f^i(q_\delta))|_{[Df^i(q_\delta)v]}\| &= \left(\frac{1-\delta}{1+\delta}\right)^{\pi(q_\delta)} \prod_{i=0}^{\pi(q_\delta)-1} \|Df(f^i(q_\delta))|_{[Df^i(q_\delta)v]}\| \\ &\leq \left(\frac{1-\delta}{1+\delta}\right)^{\pi(q_\delta)} (1+\delta)^{\pi(q_\delta)} = (1-\delta)^{\pi(q_\delta)}.\end{aligned}$$

Choose a point $x_\delta \in W^s(O(p)) \cap W^u(O(q_\delta))$ and choose a neighbourhood U of the orbit of q_δ such that:

- It does not intersect the orbit of p
- It does not intersect the future orbit of x_δ
- If the orbit of x_δ enters U it stays there for all its negative iterates by f.

Applying Gourmelon's theorem, there exists $g \in \mathcal{U}$ such that the orbit of q_δ has index greater than α and so that it preserves the α strong-stable manifold of q_δ outside U. This allow us to ensure the intersection between $W^u(O(p_g, g))$ and $W^s(O(q_\delta, g))$. Now, λ-lemma applies and we have that $q_\delta \in \overline{W^u(O(p_g, g))}$. As $H(p_g, g)$ is Lyapunov stable for f, we obtain that $\overline{W^u(O(p_g, g))} \subset H(p_g, g)$. Thus $q_\delta \in H(p_g, g)$, but this contradicts the fact that $H(p_g, g)$ is homogeneous of index α. $\square$

Proof of theorem B

First of all, we remark that once that is proved that a bi-Lyapunov stable homoclinic class $H(p, f)$ is hyperbolic then it must be the whole manifold.

Indeed, the class must be isolated. Otherwise, it cannot have local product structure. Let $\varepsilon > 0$, given by the hyperbolicity of the class, such that $W^s_\varepsilon(x) \cap W^u_\varepsilon(y)$ consists of a single point for any x sufficiently close to y in the class. Let us suppose that for some such x and y this intersection, say q, does not belong to the class. We consider $d(q, H(p, f)) > 0$ the distance of the point q to the class and define $\delta = \frac{d(q,H(p,f))}{2}$. Now, let U_δ be the δ-neighborhood of H and V the associated neighborhood given by the definition of Lyapunov stability, i.e. $f^n(V) \subset U_\delta$ for every $n \geq 1$. But, this is an absurd, since $q \in W^u_\varepsilon(y)$, there exists $N > 0$ such that $f^{-N}(q) \in V$.

Now, let U be an isolated neighborhood of the class. By bi-Lyapunov stability there exists a neighborhood $V \subset U$ of the class such that the orbits of all points in V are contained in U. As $H(p, f)$ is the maximal invariant set in U we have that $V \subset H(p, f)$. This implies that $H(p, f) = V$, in particular it is closed and open in a connected manifold, so it must be the whole manifold.

Theorem B will follows from the two propositions below.

Proposition 1. *If $H(p, f)$ is an homoclinic class for a C^1-generic diffeomorphism satisfying either the shadowing, specification or the limit shadowing property then the class is homogeneous.*

Proposition 2. *Let p be a hyperbolic periodic point, and $\Lambda = C(p, f)$ be the chain recurrent class containing p. Let $0 < \lambda < 1$ and $L \geq 1$ be given. Assume Λ satisfies the following properties:*

1. *$\Lambda = H(p, f)$ and there exists a continuous Df-invariant splitting $T_\Lambda M = E \oplus F$ with $dimE = ind(p)$ such that for every $x \in \Lambda$,*
$$||Df(x)|_{E(x)}||.||Df(x)|_{F(x)}|| < \lambda^2.$$
2. *For any $q \in Per_h(f) \cap \Lambda$, if $\pi(q) > L$ then $ind(p) = ind(q)$ and*
$$\prod_{i=0}^{\pi(q)-1} ||Df(f^i(q))|_{E(f^i(q))}|| < \lambda^{\pi(q)}$$
$$\prod_{i=0}^{\pi(q)-1} ||Df(f^{-i}(q))|_{F(f^{-i}(q))}|| < \lambda^{\pi(q)}$$

If $f|_\Lambda$ has either shadowing, or specification, or the limit shadowing property then it is hyperbolic.

We postpone the proofs of these propositions and complete the proof of theorem B. We already have property (1) since $H(p, f)$ has a dominated splitting [50]. Property (2) also holds for f. Indeed, theorem A guarantees that

$$\{Df(q), Df(f(q)), \dots, Df(f^{\pi(q)-1}(q)) : q \sim_f p\}$$

is a uniformly hyperbolic family of periodic sequences of isomorphisms of $\mathbb{R}^d$, since there are no weak eigenvalues. Hence, property (2) follows from lemma II.3 in [41]. Then, proposition 2 implies the hyperbolicity of the class. This completes the proof.

Proof of corollary C. In [51], it is proved that if f is a C^1 generic diffeomorphism then an homoclinic class with non-empty interior is bi-Lyapunov stable. We give the argument just for the sake of completeness.

Generic property (3) implies that the base point of the class can be chosen inside the interior of the class. Then the local stable and unstable manifolds are contained in the interior of the class. By invariance, the closure of stable and unstable manifolds of the point p are contained in the class. By generic property (5) we obtain $H(p, f) = \overline{W^s(p)}$ and $H(p, f) = \overline{W^u(p)}$. And then generic property (4) implies that the class is bi-Lyapunov stable.

Thus, Theorem B implies that the class is hyperbolic. □

Homogeneity of the class

In this section we prove proposition 1. First, we remark that the proposition was proved in [56] in the case that the homoclinic class has the specification property. The other cases are similar, and we prove them just for the sake of completeness.

In both cases we fix q be a periodic point in $H(p,f)$ and $\varepsilon > 0$ be such that $W^i_\varepsilon(p) \subset W^i(p)$ and $W^i_\varepsilon(q) \subset W^i(q)$, for $i = s, u$.

Shadowing

If $H(p,f)$ has the shadowing property, we proceed as follows. There exists $\delta > 0$ such that every δ-pseudo orbit is $\frac{\varepsilon}{2}$ shadowed. Let $x \in W^u(p) \cap W^s(p)$ be such that $d(x,q) < \delta$ and choose $N \in \mathbb{N}$ such that $d(f^{-N}(x), p) < \delta$. We define a sequence $(x_n)_{n\in\mathbb{N}}$ such that:

- $x_n = f^n(p)$ for all $n \leq 0$
- $x_n = f^n(f^{-N}(x))$ for all $n \in \{0, \dots, N\}$
- $x_n = f^n(f^{-N-1}(q))$ for all $n \geq N+1$

This is a δ-pseudo orbit. Hence, there is $z \in M$ that $\frac{\varepsilon}{2}$-shadows $(x_n)_{n\in\mathbb{N}}$. In particular, $z \in W^u_\varepsilon(p) \subset W^u(p)$ and $f^N(z) \in W^s_\varepsilon(q) \subset W^s(q)$. Thus $z \in W^s(q) \bigcap W^u(p)$.

By [14], we have $H(p,f) = H(q,f)$, so using a similar argument we obtain a point $z' \in W^s(p) \bigcap W^u(q)$. By Kupka-Smale's theorem, these intersections are transversal. This implies that p and q are homoclinically related and therefore have the same index.

Limit Shadowing

If the class has the limit shadowing property, consider p and q as before. Since they are in the same chain recurrent class, there exists a pseudo orbit $\{y_0, y_1, \dots, y_k\} \subset H(p,f)$ such that $x_0 = p$ and $x_k = q$. Consider the sequence $(x_n)_{n\in\mathbb{N}}$ defined by

- $x_n = f^n(p)$ for all $n \leq 0$
- $x_n = y_n$ for all $n \in \{0, 1, \dots, k\}$
- $x_n = f^n(f^{-k}(q))$ for all $n \geq K+1$

Clearly it is a limit pseudo orbit contained in $H(p,f)$. Then there exists a point $w \in M$ such that $d(f^i(w), x_i) \to 0$ if $|i| \to \infty$. A similar argument as before shows that $w \in W^u(p) \bigcap W^s(q)$.

Again, by [14] we can argument similarly and obtain a point $w' \in W^s(p) \bigcap W^u(q)$. Thus by Kupka-Smale's theorem these intersections are transversal and then p and q are homoclinically related.

Proof of Proposition 2

To prove this proposition we only need to prove that the subbundle E is contracting, the argument used to prove that F is expanding is the same. We remark that proposition 2 was proved in [24] if the class has the shadowing property. In the other cases we need to prove the following technical lemma, after that the proof goes as in [24]. In order to not complicate the notation, we will use $Df|_{E(x)}$ to denote $Df(x)|_{E(x)}$.

Lemma 2. *Let $\Lambda = C(p, f)$ satisfying all hypothesis of Proposition 2. Assume E is not contracting, then for any $\lambda < \eta < \eta' < 1$ and any neighborhood U of Λ there exists $a \in U$ with $Orb(a) \subset U$ such that*

$$\liminf_{n\to\infty} \sum_{j=0}^{n-1} \log(||Df|_{E(f^j(a))}||) < \eta < \limsup_{n\to\infty} \sum_{j=0}^{n-1} \log(||Df|_{E(f^j(a))}||) < \eta'$$

Indeed, as we will see below, in section 6, if E is not contracting, the orbit given by this lemma will have some segments that can be shadowed, creating periodic orbits that will contradict the item (2) of proposition 2.

Specification Property

In this section we prove lemma 2 when $f|_\Lambda$ has the specification property:

Suppose E is not uniformly contracted, then there exists $b \in \Lambda$ such that

$$\prod_{j=0}^{n-1} ||Df|_{E(f^j(b))}|| \geq 1 \text{ for all } n \geq 1.$$

Let

$$\alpha\colon = \min\left\{\frac{1}{6}\log\eta - \frac{1}{3}\log\eta', \frac{1}{4}(\log\eta - \log\lambda)\right\}.$$

Fix ε small enough such that if $d(x, y) < \varepsilon$, then

$$|\log(||Df|_{E(x)}||) - \log(||Df|_{E(y)}||)| < \alpha.$$

For this ε choose $L \in \mathbb{N}$ from the specification property. Let

$$K\colon = \max\{\log(||Df(x)||; x \in M)\}.$$

Let $\mathbf{Q} \subset \Lambda$ be a hyperbolic periodic orbit with $\pi(\mathbf{Q}) > L$. We will construct, by induction, a sequence of L-specifications by combining $O(b)$ with $\mathbf{Q}$. Fix $q \in \mathbf{Q}$.

By hypothesis (2) of proposition 2, we can choose $n_1 \in \mathbb{N}$ such that

$$\frac{1}{n_1\pi(q) + L}\left(\sum_{j=0}^{n_1\pi(q)-1} \log||Df|_{E(f^j(q))}|| + L\log K\right) < \frac{1}{2}(\log(\lambda) + \log(\eta)).$$

Define $e_1 = n_1\pi(q) + L$ and

$$G_1 = \sum_{j=0}^{n_1\pi(q)-1} \log ||Df|_{E(f^j(q))}|| + L\log K.$$

Then the above inequality tells that

$$\frac{1}{e_1}G_1 < \frac{1}{2}(\log(\lambda) + \log(\eta)).$$

By the choice of b we can choose $l_1 \in \mathbb{N}$ such that

$$\frac{1}{l_1}\sum_{j=0}^{l_1-1} \log ||Df|_{E(f^j(b))}|| \geq \frac{1}{2}(\log\eta' + \log\eta)$$

and if we define $f_1 = e_1 + L + l_1 = n_1\pi(q) + 2L + l_1$ and

$$B_1 = G_1 + L\log K + \sum_{j=0}^{l_1} \log ||Df|_{E(f^j(b))}||$$

then

$$\frac{1}{f_1}B_1 \geq \frac{1}{3}(\log\eta + \log\eta').$$

Define

$$I_1 = \{0, \ldots, n_1\pi(q)\} \text{ and } I_1' = \{n_1\pi(q) + L, \ldots, n_1\pi(q) + L + l_1\}.$$

If we set $P(n) = f^n(q)$ for all $n \in I_1$ and $P(n) = f^{-n_1\pi(q)-L+n}(b)$ for all $n \in I_1'$, we get a L-spaced specification $(I_1, I_1'; P)$.

Suppose we have defined $n_1, \ldots, n_k$, $l_1, \ldots, l_k$ and $I_1, I_1', \ldots, I_k, I_k'$ satisfying the following.

For every $1 \leq m \leq k$, we set $N_m\colon = \sum_{j=1}^m n_j$, $L_m\colon = \sum_{j=1}^m l_j$,

$$G_m = \sum_{i=1}^{m}\sum_{j=0}^{n_i\pi(q)-1} \log ||Df|_{E(f^j(q))}|| + 2(m-1)L\log K + \sum_{i=1}^{m-1}\sum_{j=0}^{l_i} \log ||Df|_{E(f^j(b))}||,$$

$$\begin{aligned} B_m &= G_m + L\log K + \sum_{j=0}^{l_m} \log \|Df|_{E(f^j(b)}\| \\ &= \sum_{i=1}^{m}\sum_{j=0}^{n_i\pi(q)-1} \log ||Df|_{E(f^j(q))}|| + 2mL\log K + \sum_{i=1}^{m}\sum_{j=0}^{l_i} \log ||Df|_{E(f^j(b))}||, \end{aligned}$$

$e_m\colon = N_m\pi(q) + (2m-1)L + L_{m-1}$ and $f_m\colon = e_m + L + l_m = N_m\pi(q) + (2m)L + L_m$

Then, we require that

$$\frac{1}{e_m} G_m < \frac{1}{2}(\log(\lambda) + \log(\eta)),$$
$$\frac{1}{f_m} B_m \geq \frac{1}{3}(\log(\eta) + \log(\eta')),$$

and that $(I_1, I_1', \ldots, I_m, I_m'; P)$ is a L-spaced specification.

Hypothesis (2) of proposition 2 allow us again to choose $n_{k+1} \in \mathbb{N}$ large enough such that

$$\frac{1}{n_{k+1}\pi(q) + L} \sum_{j=0}^{n_{k+1}\pi(q)-1} \log \|Df|_{E(f^j(q))}\| < \log \lambda \quad \text{and}$$

$$\frac{1}{f_k + n_{k+1}\pi(q) + L}(B_k + L\log K) \text{ is sufficiently small.}$$

We set $e_{k+1} = f_k + n_{k+1}\pi(q) + L$ and

$$G_{k+1} \colon = B_k + L\log K + \sum_{j=0}^{n_{k+1}\pi(q)-1} \log \|Df|_{E(f^j(q))}\|.$$

Thus,

$$\frac{1}{e_{k+1}} G_{k+1} < \frac{1}{2}(\log \lambda + \log \eta).$$

The choice of b allow us to choose $l_{k+1} \in \mathbb{N}$ such that:

$$\frac{1}{e_{k+1} + l_{k+1} + L} \sum_{j=0}^{l_{k+1}-1} \log \|Df|_{E(f^j(b))}\| \geq \frac{1}{2}(\log \eta + \log \eta') \text{ and}$$

$$\frac{1}{e_{k+1} + l_{k+1} + L}(G_{k+1} + L\log K) \text{ is sufficiently small}$$

We set, $f_{k+1} = e_{k+1} + l_{k+1} + L$ and

$$B_{k+1} = G_{k+1} + L\log K + \sum_{j=0}^{l_{k+1}} \|Df|_{E(f^j(b))}\|.$$

Thus,

$$\frac{1}{f_{k+1}} B_{k+1} \geq \frac{1}{3}(\log(\eta) + \log(\eta'))$$

We define

$$I_{k+1} = \{0, \ldots, n_{k+1}\pi(q)\} \text{ and } I_{k+1}' = \{n_{k+1}\pi(q) + L, \ldots, n_{k+1}\pi(q) + L + l_{k+1}\}.$$

If we set $P(n) = f^n(q)$ for all $n \in I_{k+1}$ and $P(n) = f^{-n_{k+1}\pi(q)-L+n}(b)$ for all $n \in I_{k+1}'$, then $(I_1, I_1', \ldots, I_{k+1}, I_{k+1}'; P)$ is a L-spaced specification.

By induction, we define sequences $(n_k)_{k\in\mathbb{N}}$, $(l_k)_{k\in\mathbb{N}}$ of natural numbers and $(I_1, I_1', \ldots, I_k, I_k'; P)_{k\in\mathbb{N}}$ of L-spaced specifications.

The specification property ensures the existence of a sequence $(x_k)_{k\in\mathbb{N}} \subset \Lambda$ such that x_k ε-shadows the specification $(I_1, I_1', \ldots, I_k, I_k'; P)$ for all $k \in \mathbb{N}$.

The choice of ε implies that each point x_k satisfies the following.

For each $1 \le m \le k$, if we define

$$\begin{aligned} GS_m &= \sum_{j=0}^{e_m-1} \log ||Df|_{E(f^j(x_k))}|| \\ &= \sum_{i=1}^{m}\sum_{j\in I_i} \log \|Df|_{E(f^j(x_k))}\| + \sum_{i=1}^{m-1}\sum_{j\in I_i'} \log \|Df|_{E(f^j(x_k))}\| + (2m-1)L\log K \end{aligned}$$

Then

$$\begin{aligned} GS_m &< \left(\sum_{i=1}^{m}\sum_{j\in I_i} \log \|Df|_{E(P(j))}\| + \sum_{i=1}^{m-1}\sum_{j\in I_i'} \log \|Df|_{E(P(j))}\| + (2m-1)L\log K\right) + \\ &+ \ e_m\alpha = G_m + e_m\alpha \end{aligned}$$

and

$$\frac{1}{e_m} GS_m < \frac{1}{e_m} G_m + \alpha < \frac{3}{4}\log\eta + \frac{1}{4}\log\lambda,$$

And if we define

$$\begin{aligned} BS_m &:= \sum_{j=0}^{f_m-1} \log ||Df|_{E(f^j(x_k))}|| \\ &= \sum_{i=1}^{m}\sum_{j\in I_i} \log \|Df|_{E(f^j(x_k))}\| + \sum_{i=1}^{m}\sum_{j\in I_i'} \log \|Df|_{E(f^j(x_k))}\| + 2mL\log K \end{aligned}$$

then,

$$\begin{aligned} BS_m &\ge \left(\sum_{i=1}^{m}\sum_{j\in I_i} \log \|Df|_{E(P(j))}\| + \sum_{i=1}^{m}\sum_{j\in I_i'} \log \|Df|_{E(P(j))}\| + 2mL\log K\right) - \\ &- \ f_m\alpha = B_m - f_m\alpha \end{aligned}$$

and

$$\frac{1}{f_m} BS_m \ge \frac{1}{f_m} B_m - \alpha \ge \frac{2}{3}\log\eta' + \frac{1}{6}\log\eta$$

By compactness, we may assume, by taking a subsequence if necessary, that $(x_k)_{k\in\mathbb{N}}$ converges to a point $a \in \Lambda$. We claim that the point a satisfies the lemma. Indeed, for each $N \in \mathbb{N}$, we can take $n \in \mathbb{N}$ larger enough such that the segments of orbit $\{a, \ldots, f^N(a)\}$ and $\{x_n, \ldots, f^N(x_n)\}$ are ε-close for the Hausdorff distance.

By the choice of ε we can argue similarly as above to prove that the point a satisfies:

$$\frac{1}{N}\left(\sum_{j=0}^{N-1} \log ||Df|_{E(f^j(a))}||\right) < \log(\eta), \text{ if } N \in [e_k, f_k) \text{ for some } k \in \mathbb{N}$$
$$\frac{1}{N}\left(\sum_{j=0}^{N-1} \log ||Df|_{E(f^j(a))}||\right) \geq \log(\eta'), \text{ if } N \in [f_k, e_{k+1}) \text{ for some } k \in \mathbb{N}$$

And this happens for all $N \in \mathbb{N}$. Thus we can conclude that the orbit of a satisfies the lemma.

The limit shadowing case

We prove 2 when $f|_\Lambda$ has the limit shadowing property:

If E is not uniformly contracted we can choose b as in the specification case and $\varepsilon > 0$ small enough such that if $d(x, y) < \varepsilon$, then

$$|\log(||Df|_{E(x)}||) - \log(||Df|_{E(y)}||)| < \min\left\{\frac{1}{3}(\log\eta - \log\eta'), \frac{1}{2}(\log\eta - \log\lambda)\right\}.$$

Now, we create a limit pseudo orbit as follows.

First we consider a sequence of periodic orbits $(\mathbf{Q}_n) \subset \Lambda$ such that $\mathbf{Q}_n$ is $\frac{1}{n}$-dense in Λ. We can suppose that $\pi(\mathbf{Q}_n) > L$ for all $n \in \mathbb{N}$. We then consider a point $q_1 \in \mathbf{Q}_1$ such that $d(q_1, b) < 1$. Define $x_{-j} = f^{-j}(q_1)$ for all $j \geq 0$.

Let $n_1 = 1$. Define $e'_1 = n_1\pi(q_1)$ and

$$G'_1 = \sum_{j=0}^{n_1\pi(q_1)-1} \log ||Df|_{E(x_j)}||$$

Then

$$\frac{1}{e'_1}G'_1 < \frac{1}{2}(\log\lambda + \log\eta).$$

Define $x_j = f^j(q_1)$ for $j = 1, \ldots, n_1\pi(q_1) - 1$.

Choose $l_1 \in \mathbb{N}$ such that

$$\frac{1}{l_1}\sum_{j=0}^{l_1-1} \log \|Df|_{E(f^j(b))}\| \geq \frac{1}{2}(\log\eta + \log\eta')$$

and if $f'_1 = e'_1 + l_1$ and

$$B'_1 = \sum_{j=0}^{n_1\pi(q_1)-1} \log ||Df|_{E(x_j)}|| + \sum_{j=0}^{l_1-1} \log \|Df|_{E(f^j(b))}\|,$$

then

$$\frac{1}{f'_1} B'_1 \geq \frac{1}{3}(\log(\eta) + \log(\eta'))$$

Define $x_j = f^j(f^{-n_1\pi(q_1)}(b))$ for $j = n_1\pi(q_1), \ldots, n_1\pi(q_1) + l_1 - 1$.

Suppose we have defined $q_1, \ldots, q_k$, $n_1, \ldots, n_k$, $l_1, \ldots, l_k$ and $(x_j)_{j=-\infty}^{N_k+L_k}$ satisfying the following.

For every $1 \leq m \leq k$, $d(q_m, (b)) < \frac{1}{m}$. We define $N_m\colon = \sum_{j=1}^{m} n_j\pi(q_j)$, $L_m\colon = \sum_{j=1}^{m} l_j$,

$$G'_m = \sum_{i=0}^{m} \sum_{j=0}^{n_i\pi(q_i)-1} \log ||Df|_{E(f^j(q_j))}|| + \sum_{i=0}^{m-1} \sum_{j=0}^{l_i-1} \log ||Df|_{E(f^j(b))}\|,$$

$$B'_m = \sum_{i=0}^{m} \sum_{j=0}^{n_i\pi(q_i)-1} \log ||Df|_{E(f^j(q_j))}|| + \sum_{i=0}^{m} \sum_{j=0}^{l_i-1} \log ||Df|_{E(f^j(b))}\|,$$

$e'_m = N_m + L_{m-1}$ and $f'_m = N_m + L_m$.

We require that

$$\frac{1}{e'_m} G'_m < \frac{1}{2}(\log\lambda + \log\eta)$$

and that

$$\frac{1}{f'_m} B'_m \geq \frac{1}{3}(\log\eta + \log\eta')$$

Also define

$$x_j = \begin{cases} f^j(f^{-N_{m-1}-L_{m-1}}(q_m)) & \text{if } j \in [f'_{m-1}; e'_m) \\ f^j(f^{-N_m-L_{m-1}}(b)) & \text{if } j \in [e'_m; f'_m) \end{cases}.$$

Let $q_{k+1} \in \mathbf{Q}_{k+1}$ such that $d(q_{k+1}, b)) < \frac{1}{k+1}$.

Hypothesis (2) allows us to choose $n_{k+1} \in \mathbb{N}$ such that:

$$\frac{1}{n_{k+1}\pi(q_{k+1})} \sum_{j=0}^{n_{k+1}\pi(q)-1} \log \|Df|_{E(f^j(q_{k+1}))}\| < \log\lambda \text{ and}$$

$$\frac{1}{f'_k + n_{k+1}\pi(q_{k+1})} \left(\sum_{i=0}^{m} \sum_{j=0}^{l_i-1} \log ||Df|_{E(f^j(b))}|| \right) \text{ is sufficiently small.}$$

If we define

$$G'_{k+1} = \sum_{i=0}^{k+1} \sum_{j=0}^{n_i\pi(q_i)-1} \log ||Df|_{E(f^j(q_j))}|| + \sum_{i=0}^{k} \sum_{j=0}^{l_i-1} \log ||Df|_{E(f^j(b))}||$$

then

$$\frac{1}{e'_{k+1}} G'_{k+1} < \frac{1}{2}(\log\lambda + \log\eta)$$

The choice of b allow us to choose $l_{k+1} \in \mathbb{N}$ such that:

$$\frac{1}{e'_{k+1} + l_{k+1}} \sum_{j=0}^{l_{k+1}-1} \log \|Df|_{E(f^j(b))}\| \geq \frac{1}{2}(\log\eta + \log\eta') \text{ and}$$

$$\frac{1}{e'_{k+1} + l_{k+1}} G'_{k+1} \text{ is sufficiently small.}$$

If we define $f'_{k+1} = e'_{k+1} + l_{k+1}$ and

$$B'_{k+1} = \sum_{i=0}^{k+1} \sum_{j=0}^{n_i\pi(q_i)-1} \log ||Df|_{E(f^j(q_j))}|| + \sum_{i=0}^{k+1} \sum_{j=0}^{l_i-1} \log ||Df|_{E(f^j(b))}||,$$

then

$$\frac{1}{f'_{k+1}} B'_{k+1} \geq \frac{1}{3}(\log\eta + \log\eta')$$

Thus, by induction we define sequences $(n_k)_{k\in\mathbb{N}}$, $(l_k)_{k\in\mathbb{N}}$ of natural numbers, $(q_k)_{k\in\mathbb{N}}$, $q_k \in \mathbf{Q}_k$, and $(x_j)_{j\in\mathbb{Z}} \subset \Lambda$ satisfying the above inequalities.

By the choice of $(q_k)_{k\in\mathbb{N}}$, $(x_j)_{j\in\mathbb{Z}}$ is a limit pseudo orbit contained in Λ that satisfies the following.

For each $k \geq 1$,

$$\frac{1}{e'_k} \sum_{i=0}^{e'_k-1} \log ||Df|_{E(x_i)}|| < \frac{1}{2}(\log\lambda + \log\eta),$$

$$\frac{1}{f'_k} \sum_{i=0}^{f'_k-1} \log ||Df|_{E(x_i)}|| \geq \frac{1}{3}(\log\eta + \log\eta')$$

Since f has the limit shadowing property, there exists a point $y \in M$ such that $d(f^i(y), x_i) \to 0$ when $|i| \to \infty$. Then, there exists $n \in \mathbb{N}$ such that $d(f^i(y), x_i) < \varepsilon$ for all $i \geq n$. By the choice of ε, we can argue similarly to the specification case and obtain that the point $a = f^n(y)$ satisfies the lemma. □

End of the proof of Proposition 2

Using Proposition 2.3 and lemma 3.7 in [24] if we suppose that the bundle E does not contract, we can obtain periodic orbits $(q_k)_{k\in\mathbb{N}}$ arbitrarily close to Λ such that their stable and unstable manifold have an uniform size, that is, there exists $\varepsilon > 0$ such that $W^{cs}_{\varepsilon}(q_k) \subset W^s(q_k)$ and $W^{cu}_{\varepsilon}(q_k) \subset W^u(q_k)$, where W^{cs} denotes the local foliation tangent to the subbundle E of the dominated splitting $E \oplus F$ over the homoclinic class (the same for W^{cu}), see [32]. Moreover, we have:

$$\prod_{i=0}^{n-1} ||Df|_{E(f^i(q_k))}|| < \eta'^n \ \ for\ any\ \ n \geq 1$$

$$\prod_{i=0}^{n-1} ||Df^{-1}|_{F(f^{-i}(q_k))}|| \geq \left(\frac{\lambda^2}{\eta}\right)^n \ \ for\ any\ \ n \geq 1.$$

This implies that if k is large enough, all q_i, $i \geq k$, are homoclinically related, that is $H(q_i, f) = H(q_j, f)$ for any i,$j \geq k$. By taking a subsequence, we may assume that $(q_k)_{k\in\mathbb{N}}$ converge to a point $z \in H(p, f)$. Therefore, $q_i \in H(p, f)$ for any $i \geq k$. But this contradicts property (2) of Proposition 2. Hence we obtain that the bundle E must contract uniformly. We can make the same argument for the bundle F and obtain hyperbolicity of the class. This completes the proof of Proposition 2.

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