UNIVERSIDADE FEDERAL DO RIO DE JANEIRO
INSTITUTO DE MATEMÁTICA
INSTITUTO TÉRCIO PACITTI DE APLICAÇÕES E PESQUISAS COMPUTACIONAIS
PROGRAMA DE PÓS-GRADUAÇÃO EM INFORMÁTICA

JULIANA CASTANON XAVIER

NUMERICAL ANALYSIS OF BOUSSINESQ SYSTEMS

Rio de Janeiro
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A thesis submitted to the faculty of the Department of Computer Science of the Institute of Mathematics and the Institute Tércio Pacitti of Applications and Computational Research of Federal University of Rio de Janeiro, in partial fulfillment of the requirements for the degree of Doctor in Computer Science.

Advisor: Mauro Antonio Rincon

Co-advisor: Daniel Gregorio Alfaro Vigo

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To my parents, Iolanda and Joel,
my brother Joel and
my husband Rodrigo...

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Even though this thesis has been written in English, the acknowledgments will be written in Portuguese.

Mesmo sendo clichê, não existe outra maneira de começar a escrever meus agradecimentos: obrigada, Deus! Obrigada por cada segundo que não me deixaste desistir; por me ajudar a ter a dedicação necessária e de alguma forma sempre me fazer ter forças para continuar. Obrigada por me permitir estar escrevendo essa página de agradecimento. Ao Senhor, minha gratidão eterna.

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RESUMO

Xavier, Juliana Castanon. **Numerical analysis of Boussinesq systems**. 2016. 139 f. Thesis (Doctor in Computer Science) - PPGI, Instituto de Matemática, Instituto Tércio Pacitti de Aplicações e Pesquisas Computacionais, Universidade Federal do Rio de Janeiro, Rio de Janeiro, 2016.

Neste trabalho estudamos numericamente os sistemas de Boussinesq. Inicialmente, apresentamos a análise de estabilidade da família linear de sistemas de Boussinesq com o objetivo de determinar a influência de seus parâmetros na eficiência e precisão do método espectral de colocação de Fourier aplicado na variável espacial, juntamente com o método de Runge Kutta de quarta ordem aplicado na variável temporal. São identificadas quais regiões de parâmetros são as mais adequadas para a obtenção de uma solução numérica consistente. Na sequência, apresentamos a análise de convergência da família não linear de sistemas de Boussinesq nos casos em que a condição de estabilidade linear é dada por $\Delta t < C\Delta x$. Essa análise nos fornece estimativas de erro no espaço e no tempo para o cálculo da solução aproximada. Experimentos numéricos são fornecidos com o objetivo de validar o código implementado para a determinação da solução aproximada, verificar a estabilidade das soluções do problema linear nas regiões de parâmetros que apresentam resolução numérica com baixo custo computacional, bem como comprovar a ordem de convergência esperada da solução numérica para o problema não linear. Também são mostrados diversos experimentos referentes ao estudo numérico de ondas solitárias para esses sistemas.

Palavras-chave: Sistemas de Boussinesq, Análise de Estabilidade, Análise de Convergência, Método de Colocação de Fourier, Método de Runge Kutta, Simulações Numéricas.

ABSTRACT

Xavier, Juliana Castanon. **Numerical analysis of Boussinesq systems**. 2016. 139 f. Thesis (Doctor in Computer Science) - PPGI, Instituto de Matemática, Instituto Tércio Pacitti, Universidade Federal do Rio de Janeiro, Rio de Janeiro, 2016.

In this thesis we perform the numerical analysis of the family of Boussinesq systems. Initially, the stability analysis of the linear family of these systems leads us to determine the influence of its parameters on the efficiency and accuracy of the numerical scheme. As a consequence of this analysis, we identify which regions of parameter space are most appropriate for obtaining a consistent numerical solution. In sequence, we apply the convergence analysis to the nonlinear family of Boussinesq system with stability condition of the type $\Delta t < C\Delta x$ in order to prove H^s-error bounds of spectral accuracy in space and of fourth-order accuracy in time. The systems are discretized in space by the standard Fourier collocation spectral method and in time by the explicit fourth order Runge-Kutta (RK4) scheme. Numerical experiments are shown in order to validate the accuracy of our numerical method, and verify the stability of the linear solution in each region of parameters as well as the order of convergence of the numerical method applied in these systems. Moreover, we also do some comments about what happens with the stability of the solution for the nonlinear problem and study numerically the solitary waves for these systems.

Keywords: Boussinesq Systems, Stability Analysis, Convergence Analysis, Fourier Collocation Method, Runge Kutta Method, Numerical Simulations.

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1 INTRODUCTION

The general aim of this thesis is to perform the numerical analysis of surface waves propagation in a fluid with a well-defined volume. For an ideal fluid, which means an inviscid, incompressible and irrotational fluid, the 2D Euler equations in a continuous approximation, describes the motion of a free surface over a horizontal bottom at height $y = -h_0$ by

$$u_t + uu_x + vu_y + \frac{1}{\rho}p_x = 0, \tag{1.1}$$

$$v_t + uv_x + vv_y + \frac{1}{\rho}p_y = -g, \tag{1.2}$$

$$u_x + v_y = 0, \tag{1.3}$$

$$u_y = v_x, \tag{1.4}$$

which holds for all $t \geq 0$ and $(x, y) \in \Omega_t = \{(x, y) \mid x \in \mathbb{R},\ -h_0 \leq y \leq \eta(x, t)\}$, $h_0 > 0$. The function $\eta(x, t)$ indicates the deviation of the free surface of the fluid above its level of rest, g is the acceleration of gravity, $u = u(x, y, t)$ and $v = v(x, y, t)$ denote, respectively, the horizontal and vertical velocity components, ρ is the constant density and $p = p(x, y, t)$ is the pressure.

The system (1.1)-(1.4) is supplemented by the free surface kinematic and dynamic boundary conditions

$$\eta_t + u\eta_x = v \quad \text{at} \quad y = \eta(x, t), \tag{1.5}$$

$$p = 0 \quad \text{at} \quad y = \eta(x, t). \tag{1.6}$$

At the bottom, we assume that

$$v = 0 \quad \text{at} \quad y = -h_0. \tag{1.7}$$

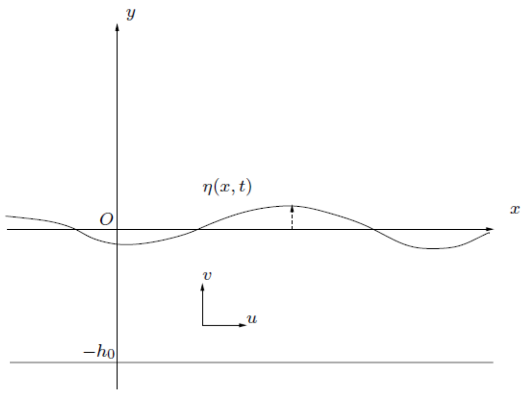

Figure 1.1: 2D Euler equations variables. Source: KAMPANIS; DOUGALIS; EKATERINARIS (2008)

We also assume that initial conditions for η and u have been specified, and let

$$\eta(x,0) = \Phi(x) \quad \text{for} \quad x \in \mathbb{R}, \tag{1.8}$$

$$u(x,y,0) = \Psi(x,y) \quad \text{for} \quad (x,y) \in \Omega_0, \tag{1.9}$$

where $\Phi(x)$ and $\Psi(x)$ are given functions.

It happens that for some engineering applications, the full system (1.1)-(1.9) has been more difficult than it should be in some situations. For that reason, it has been derived several asymptotic models from the Euler equations with some physical restrictions.

A regime that arises in practical situations, is the one of waves in a channel of approximately constant depth h_0 that are uniform across the channel, which are of small amplitude and long wavelength, and such that the associated nonlinear and dispersive effects are balanced. If A denotes a typical wave amplitude and λ a typical wavelength, the conditions just mentioned can be expressed as

$$\frac{A}{h_0} \ll 1, \quad \frac{h_0^2}{\lambda^2} \ll 1, \quad \frac{A\lambda^2}{h_0^2} \approx 1. \tag{1.10}$$

In 1870, Boussinesq derived the first models for evolution equations. These models were applicable, in principle, to describe motions that are two-dimensional and which have the form of a perturbation of the one-dimensional wave equation (see BOUSSINESQ (1872)). Moreover, these equations were derived directly from the Eulerian formulation of the water wave problem using the assumption, among others, that the waves travel only in one direction. As a consequence of this assumption, these equations are formally comparable to the one derived by Korteweg and de-Vries in KORTEWEG; VRIES (1895), the well-known Korteweg-de Vries equation, or in other words, the KdV equation.

It is worth noting that Boussinesq in BOUSSINESQ (1871), also derived from the Euler equations a system of two coupled equations,

$$\begin{aligned} \eta_t + u_x + (u\eta)_x &= 0, \\ u_t + \eta_x + uu_x + \tfrac{1}{3}u_{xxt} &= 0, \end{aligned} \tag{1.11}$$

which are free of the assumption of unidirectionality, the main characteristic for equations that models surface-wave propagation, similar to the first equations obtained by him and the KdV equation.

One therefore expects that Boussinesq systems (1.11) will have more interest than the unidirectional models, because of their wider range of potential of applicability.

As with unidirectional models, there are many different but formally equivalent Boussinesq systems, as explained in BONA; CHEN; SAUT (2002). However, despite their formal equivalence as models for small-amplitude long waves, these systems may have rather different mathematical properties.

Our principal aim in this work is to examine some of the theoretical and numerical properties of Boussinesq systems of the form

$$\begin{aligned} \eta_t + u_x + (u\eta)_x + a u_{xxx} - b\eta_{xxt} = 0, \\ u_t + \eta_x + u u_x + c\eta_{xxx} - d u_{xxt} = 0, \end{aligned} \tag{1.12}$$

which are first-order approximations of the Euler equations (1.1)-(1.9), with respect to small parameters α and β introduced in (1.10), such that

$$\begin{aligned} a = \tfrac{1}{2}\Big(\theta^2 - \tfrac{1}{3}\Big)\lambda, \quad b = \tfrac{1}{2}\Big(\theta^2 - \tfrac{1}{3}\Big)(1-\lambda), \\ c = \tfrac{1}{2}\Big(1 - \theta^2\Big)\mu, \quad d = \tfrac{1}{2}\Big(1 - \theta^2\Big)(1-\mu), \end{aligned} \tag{1.13}$$

with

$$a + b + c + d = \frac{1}{3}, \tag{1.14}$$

λ, $\mu \in \mathbb{R}$ and $0 \leq \theta \leq 1$.

It is worth to observe that different choices of λ, μ and θ give rise to different parameters a, b, c, d. Therefore, the equations (1.12) denote a family of systems.

The independent variables x and t indicate the position of a particle of the fluid along the channel and time, respectively. The dependent variables $\eta = \eta(x,t)$ and $u = u(x,t)$ evaluated in each point (x,t) indicate, respectively, the deviation of the free surface of the fluid above its level of rest and the horizontal velocity component in some point above the bottom of the channel.

In BONA; CHEN; SAUT (2002) and BONA; CHEN; SAUT (2004), Bona et al. introduced and analyzed several types of Boussinesq systems arising from (1.12). Moreover, the authors raised important issues about the numerical resolution of such systems, mainly because in general, the initial-boundary-value problems(IVPs) associated with these systems do not present a analitic solution determined.

After these papers, many works appeared in this field. The numerical analysis of some of those systems have been carried out in ANTONOPOULOS; DOUGA-

LIS (2010), ANTONOPOULOS; DOUGALIS (2012a), ANTONOPOULOS; DOUGALIS (2012b), ANTONOPOULOS; DOUGALIS; MITSOTAKIS (2010a), BONA; DOUGALIS; MITSOTAKIS (2007) and DOUGALIS; MITSOTAKIS; SAUT (2010). These papers addressed a very important question raised in BONA; CHEN; SAUT (2002) concerning the construction of accurate, efficient numerical schemes for approximating solutions of interesting IVPs related to those systems. Nonetheless, it would be useful to explore in a more consistent way the properties of those Boussinesq type systems as a family in the construction of efficient numerical schemes.

Our purpose in this thesis is to identify the influence of the parameters (1.13) of the Boussinesq systems (1.12) on the efficiency and accuracy of a numerical scheme. With this aim in mind, we propose the development of numerical schemes based on the spectral Fourier collocation method for the spatial discretization along with an explicit fourth-order Runge Kutta time discretization.

The convergence analysis related to the spatial semi discretization will indicate how accurate in space we can obtain approximate solutions. This is expected to be strongly related to the regularity of the solutions of (1.12). On the other hand, the analysis of the time discretization should indicate how we can do an efficient and accurate time approximation. This information is not directly related to the space regularity of the solutions, but to the dispersive and stability properties of the time discretized equations.

Therefore, after carrying out the full numerical analysis, we shall identify which particular systems, or in other words, which parameter regions, are best suited for an accurate and efficient numerical solution. As an important application of this analysis we shall carry out the numerical simulations of solitary wave propagation and interaction.

This text have 6 chapters, which are organized as follows: in chapter 2 we give some notations, definitions and important results that are used in the following chapters; in chapter 3 we give an overview of the state of art of the analysis of Boussinesq systems, starting with their deduction from Euler equations, giving some examples and the most import results about well-posedness that are found in the literature. In chapter 4, we perform the numerical analysis of these systems, starting with the stability analysis of the linear family of Boussinesq systems, followed by the convergence analysis of the nonlinear family of Boussinesq systems. In chapter 5 we give some numerical simulations in order to validate our numerical method and to testify the results obtained during the numerical analysis. We finish in chapter 6 giving the conclusions.

2 BACKGROUND

In this chapter we give the main results contained in the literature with respect to the spectral methods and methods for solving numerically systems of ordinary differential equations (ODEs). We also cover some results about Sobolev spaces and give some important inequalities, both necessary during the convergence analysis in the chapter 4,

2.1 Spectral Methods

The spectral methods are one of the three major methods for solving numerically differential equation, along with the finite element method (FEM) and finite difference method (FDM). As a consequence, these three methods have large applications in areas such as fluid mechanics, quantum mechanics, wave phenomena, complex analysis, and so on.

Spectral methods are named in this way because they are based on the spectrum of a function, i.e., the values of its Fourier transform. They are a class of spatial discretizations for differential equations. The key components for their formulation are the *trial functions*, also called the expansion or approximating functions, and the *test functions*, also known as weight functions. The test functions are used to ensure that the differential equation and perhaps some boundary conditions are satisfied as closely as possible by the truncated series expansion. This is achieved by minimizing, with respect to a suitable norm, the residual produced by using the truncated expansion instead of the exact solution.

The choice of the trial functions is one of the features that distinguishes the early versions of spectral methods from finite-element and finite difference methods. The trial basis functions, for what can now be called *classical spectral methods*, are global, infinitely differentiable and nearly orthogonal, i.e., the matrix consisting of their inner products has very small bandwidth; in many cases this matrix is diagonal.

In contrast, for the h version of FEM, the domain is divided into small elements, and low-order trial functions are specified in each element. The trial basis functions for FEM are thus local in character and still nearly orthogonal, but not infinitely differentiable. They are thus well suited for handling complex geometries. The FE methods are typically viewed from a pointwise approximation perspective rather than from a trial function/test function perspective. However, when appropriately translated into a trial function/test function formulation, the finite-difference trial basis functions are likewise local.

Therefore, for solving differential equations with smooth initial data in a simple domain with high accuracy, in general the spectral methods show better results than the FEM or FDM. The spectral methods can reach a 10 digits precision in some cases, against 2 or 3 digits from the FEM or FDM. Moreover, for lower accuracy, spectral methods require less computational cost than the other two methods (see CANUTO et al. (2006) for details).

Another advantage of the spectral methods is the possibility of choosing the trial functions. The most frequent are the trigonometric polynomial functions, Chebyshev polynomials and Legendre polynomials. The choice of test functions distinguishes between the three earliest types of spectral schemes, namely, the Galerkin, collocation, and tau versions. In this text, we only detail the collocation method and give an example of its application in section 2.3. More details about other spectral methods can be consulted in CANUTO et al. (2006).

As we mentioned that the spectral methods are essentially based on the concepts of the Fourier transform. Therefore, before we define the spectral Fourier collocation method, we give in the following section some of the most important results about this theory.

2.2 Fourier Analysis

The expansion of a function u in terms of an infinite sequence of orthogonal functions ϕ_k, e.g., $u = \sum_{k=-\infty}^{\infty} \hat{u}_k \phi_k$ or $u = \sum_{k=0}^{\infty} \hat{u}_k \phi_k$, underlies many numerical methods of approximation. The accuracy of the approximations and the efficiency of their implementation influence decisively the domain of applicability of these methods in scientific computations.

The expansion in terms of an orthogonal system introduces a linear transformation between u and the sequence of its expansion coefficients $\hat{u}_k$. This is usually called the transform of u between physical space (space for variable x) and Fourier space (space for the wave number k). If the system is complete in a suitable Hilbert space, this transform can be inverted. Hence, functions can be described both through their values in physical space and through their coefficients in Fourier space.

The expansion coefficients depend on (almost) all the values of u in physical space, and they can rarely be computed exactly. A finite number of approximate expansion coefficients can be easily computed using the values of u at a finite number of selected points, usually the nodes of high-precision quadrature formulas. This procedure defines a *discrete transform* between the set of values of u at the quadrature points and the set of approximate, or *discrete*, coefficients. With a proper choice of the quadrature formulas, the finite series defined by the discrete transform is actually the interpolant of u at the quadrature nodes.

In the following sections, we give an overview of the main results about the continuous and discrete Fourier expansions.

2.2.1 Continuous Fourier Expansion

The continuous Fourier expansion, or the *Fourier transform* of a function $u(x)$, $x \in \mathbb{R}$ is the function $\hat{u}(k)$ defined by

$$\hat{u}_k = \int_{-\infty}^{\infty} u(x)e^{-ikx}dx, \quad k \in \mathbb{R}. \tag{2.1}$$

The number $\hat{u}(k)$ can be interpreted as the amplitude density of u at wavenumber k. Conversely, we can reconstruct u from $\hat{u}$ by the *inverse Fourier transform*

$$u(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{u}(k)e^{ikx}dk, \quad x \in \mathbb{R}. \tag{2.2}$$

From now on, the variable x is called the physical variable and k is called the Fourier variable or wavenumber. The integrals in (2.1) and (2.2) are defined in the sense of Lebesgue.

The well-known results about the continuous Fourier expansion as its properties, convergence theorems and convolution forms can be found in CANUTO et al. (2006), TREFETHEN (2000), IÓRIO; IÓRIO (2001) and KREYSZIG (1978). Since the aim of this thesis is concentrated in the numerical analysis area, and we have to use computers to simulate experiments, we are limited to perform all the calculations considering discrete data in a finite quantity. Because of that, we introduce in the following sections the most appropriate versions of the Fourier transform to complete this goal.

2.2.2 Discrete Fourier Expansion

In this section, we start considering x ranging over

$$h\mathbb{Z} = \{x_j = jh \in \mathbb{R}, j \in \mathbb{Z}\},$$

rather than $\mathbb{R}$.

Precise analogues of the Fourier transform and its inverse exist for this case. The crucial point is that because the spatial domain is discrete, the wavenumber k will no longer range over all of $\mathbb{R}$. Instead, the appropriate wavenumber domain is a bounded interval of length $2\pi/h$, and one suitable choice is $[-\pi/h, \pi/h]$. It is worth noticing that k is bounded because x is discrete; in fact,

$$\begin{array}{ccccccc} \text{Physical space} & : & \text{discrete,} & \text{unbounded} & : & x \in h\mathbb{Z} \\ & & \updownarrow & \updownarrow & & \\ \text{Fourier space} & : & \text{bounded,} & \text{continuous} & : & k \in [-\pi/h, \pi/h] \end{array}$$

The reason for these connections is the phenomenon known as *aliasing*, which means that for any complex exponential e^{ikx}, there are infinitely many other complex exponentials that match it on the grid $h\mathbb{Z}$. Consequently it suffices to measure wavenumbers for the grid in an interval of length $2\pi/h$, and for reasons of symmetry, we choose the $[-\pi/h, \pi/h]$. This fact is illustrated in the next figure.

For a function v defined on $h\mathbb{Z}$ with value v_j at x_j , the *semidiscrete Fourier transform* is defined by

$$\hat{v}(k) = h \sum_{j=-\infty}^{infty} e^{-ikx_j} v_j, \quad k \in [-\pi/h, \pi/h], \tag{2.3}$$

and the *inverse semidiscrete Fourier transform* by

$$v_j = \frac{1}{2\pi} \int_{-\pi/h}^{\pi/h} e^{ikx_j} \hat{v}(k) dk, \quad j \in \mathbb{Z}. \tag{2.4}$$

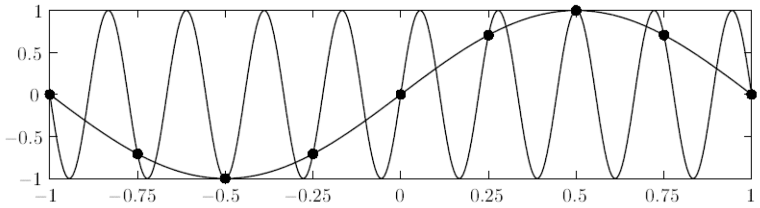

Figure 2.1: An example of aliasing. On the grid $1/4\mathbb{Z}$, the functions $\sin(\pi x)$ and $\sin(9\pi x)$ are identical. Source: TREFETHEN (2000)

The equation (2.3) approximates (2.1) by a trapezoidal rule, and (2.4) approximates (2.2) by truncating $\mathbb{R}$ to $[-\pi/h, \pi/h]$. As $h \to 0$, the two pair of formulas converge (see TREFETHEN (2000)).

In fact, the expression "semidiscrete Fourier transform" is unfamiliar, and it is being used here to emphasize that our concern is in the case where the variable x is discrete and the Fourier variable k is a bounded interval. This is the inverse problem compared to the Fourier series, which represents a function on a bounded interval as a sum of complex exponentials at discrete wavenumbers. However, mathematically, there is no difference from the theory of Fourier series.

Here, the semidiscrete Fourier transform is used to obtain an unique interpolant that is band-limited to wavenumbers in the interval $[-\pi/h, \pi/h]$. This interpolant is used to derive the spectral differentiation theory, which has as main result that if u is a differentiable function with Fourier transform $\hat{u}$, then the Fourier transform of u' is given by

$$\hat{u'}(k) = ik\hat{u}(k). \tag{2.5}$$

Observe that using the equation (2.5), one can recover the function u' applying the inverse Fourier transform. More details about the derivation of (2.5) can be found in TREFETHEN (2000).

We pass now to the concept of the *discrete Fourier transform* (DFT). The biggest gain of this formulation is that it can be computed by the *Fast Fourier Transform* (FFT), which will be detailed in the following.

We consider fully discretized periodic functions evaluated in finite quantity of points, with period 2π. At first sight, the requirement of periodicity may suggest that this method has limited relevance for practical problems. Yet periodic grids are surprisingly useful in practice. Often in scientific computing a phenomenon is of interest that is unrelated to boundaries, such as the interaction of solitons in the KdV equation (see KORTEWEG; VRIES (1895)). For such problems, periodic boundary conditions often prove the best choice for computation. In addition, some geometries are physically periodic, such as crystal lattices or rows of turbine blades. Finally, even if the physics is not periodic, the coordinate space may be, as is the case for a θ or ϕ variable in a computation involving polar or spherical coordinates (see chapter 11 of TREFETHEN (2000)).

We consider a periodic grid as a subset of the interval $[0, 2\pi]$. Note that, when we say periodic grid, we mean that any data values on the grid come from evaluating a periodic function. We may regard the periodic grid as one cycle of length N, extracted from an infinite grid with data satisfying $v_{j+mN} = v_j$ for all $j, m \in \mathbb{Z}$. We consider that N is always an even number.

The space step h is calculated as $h = 2\pi/N$, which implies that $\dfrac{\pi}{h} = \dfrac{N}{2}$. Therefore, here the Fourier domain is discrete as well as bounded. This is because waves in physical space must be periodic over the interval $[0, 2\pi]$, and only waves e^{ikx} with integer wavenumbers have the required period 2π. Then, we find

$$\begin{array}{ccccccc}
\text{Physical space} & : & \text{discrete,} & \text{bounded} & : & x \in \{h, 2h, \ldots, 2\pi - h, 2\pi\} \\
 & & \updownarrow & \updownarrow & & \\
\text{Fourier space} & : & \text{bounded,} & \text{discrete} & : & k \in \{-\frac{N}{2}+1, -\frac{N}{2}+2, \ldots, \frac{N}{2}\}
\end{array}$$

The formula for the DFT is

$$\hat{v}_k = h \sum_{j=1}^{N} e^{-ikx_j} v_j, \quad k = -\frac{N}{2} + 1, \ldots, \frac{N}{2}, \tag{2.6}$$

and the *inverse discrete Fourier transform* is given by

$$v_j = \frac{1}{2\pi} \sum_{k=-N/2+1}^{N/2} e^{ikx_j} \hat{v}_k, \quad j = 1, \ldots, N. \tag{2.7}$$

The spectral differentiation in this formulation follows the same idea for the semidiscrete Fourier transform mentioned before. However, in this case we have a complication to address: evaluating the inverse transform (2.7) as it stands would give a term $e^{iNx/2}$ with derivative $(iN/2)e^{iNx/2}$. Since $e^{iNx/2}$ represents a real, sawtooth wave on the grid, its derivative should be zero at the grid points, not a complex exponential. The problem is that (2.7) treats the highest wavenumber asymmetrically. We can fix this by defining $\hat{v}_{-N/2} = \hat{v}_{N/2}$ and replacing (2.7)

$$v_j = \frac{1}{2\pi} \sideset{}{'}\sum_{k=-N/2+1}^{N/2} e^{ikx_j} \hat{v}_k, \quad j = 1, \ldots, N, \tag{2.8}$$

where the prime indicates that the terms $K = \pm N/2$ are multiplied by $\frac{1}{2}$.

The spectral differentiation can be summed as: given a function v, we compute its DFT $\hat{v}$; then, we define $\hat{w}_k = ik\hat{v}_k$, taking special care of the $\hat{w}_{N/2}$ term. In sequence, compute w from $\hat{w}$. The problematic term associated with the wavenumber $k = N/2$ generates a loss of symmetry for odd derivatives, and we have to set $\hat{w}_{N/2} = 0$ in order to compute higher derivatives.

In fact, to approximate the νth derivative, we do:

- Given v, compute $\hat{v}$;

- Define $\hat{w}_k = (ik)^\nu \hat{v}_k$, setting $\hat{w}_{N/2} = 0$ if ν is odd;
- Compute w from $\hat{w}$.

The computation of the discrete Fourier transform can be accomplished by the Fast Fourier Transform (FFT), discovered in 1965 by Cooley and Tukey. If N is highly composite, that is, a product of small prime factors, then the Fast Fourier Transform enables us to compute the discrete Fourier transform, and hence spectral derivatives, in $O(N \log N)$ floating point operations.

It is worth noticing that spectral derivatives obtained by differentiation matrix, which means that the formulas (2.6) and (2.7) were used directly, are calculated in $O(N^2)$ floating points operations. This comparison proves the real numerical gain in using the FFT (see CANUTO et al. (2006) and TREFETHEN (2000) for details).

2.3 Fourier Collocation Method - Formulation and Application

In the collocation approach the test functions are translated Dirac delta-functions centered at special, so-called *collocation points*. This approach requires the differential equation to be satisfied exactly at the collocation points. In the following we illustrate the basic principles of the collocation method and the basic properties of the set of polynomials chosen as trial functions.

We consider the nonlinear Burguers equation, derived in 1948 as a simplified model of the Navier-Stokes equation, given by

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} - \nu\frac{\partial^2 u}{\partial x^2} = 0, \tag{2.9}$$

for all $t > 0$, considering $\Omega = [0, 2\pi]$ and ν is a positive constant. We supplement this equation with the initial condition $u(x,0) = u_0(x)$ in Ω.

We seek for a function u_N, which is considered an approximate solution in Ω of the equation (2.9), obtained through the collocation Fourier method.

We consider the trial space $S_N = \text{span}\left\{e^{-ikx} : -N \leq k \leq N\right\}$, the set of all trigonometric polynomials of degree $\leq N/2$. The set of functions S_N is an orthogonal system over the interval $(0, 2\pi)$, such that

$$\int_0^{2\pi} e^{-ikx}\overline{e^{-ilx}}dx = 2\pi\delta_{kl} = \begin{cases} 0 & \text{if} \quad k \neq l, \\ 2\pi & \text{if} \quad k = l. \end{cases}$$

(The overline on e^{-ilx} denotes its complex conjugate.)

The approximate solution u_N is represented by its values at the grid points $x_j = 2\pi j/N$, $j = 0, \ldots, N-1$. In turn, the grid point values of u_N are related to its discrete Fourier coefficients (defined in section 2.2.2) by

$$\hat{u}_k = \frac{1}{N}\sum_{j=1}^{N} u(x_j)e^{-ikx_j}, \quad k = -N/2+1, \ldots, N/2, \tag{2.10}$$

and

$$u(x_j) = \sum_{k=-N/2+1}^{N/2} \hat{u}_k e^{ikx_j}, \quad j = 1, \ldots, N. \tag{2.11}$$

For the collocation method we require that the respective system be satisfied at these grid points, that is,

$$\left.\frac{\partial u_N}{\partial t} + u_N\frac{\partial u_N}{\partial x} - \nu\frac{\partial^2 u^N}{\partial x^2}\right|_{x=x_j} = 0, \ j = 0, 1, \ldots, N-1, \tag{2.12}$$

with initial condition at each collocation point given by $u_N(x_j, 0) = u_0(x_j)$.

Taking $\mathbf{u}(t) = (u_N(x_0,t), u_N(x_1,t), \ldots, u_N(x_{N-1},t))^T$, we can write (2.12) in a vector form as

$$\frac{d\mathbf{u}}{dt} + \mathbf{u} \otimes D_N\mathbf{u} - \nu D_N^2\mathbf{u} = 0, \tag{2.13}$$

or

$$\frac{d\mathbf{u}}{dt} + \frac{1}{2}D_N(\mathbf{u} \otimes \mathbf{u}) - \nu D_N^2\mathbf{u} = 0, \tag{2.14}$$

where $\mathbf{u} \otimes \mathbf{v}$ indicates each component of the product between $\mathbf{u}$ and $\mathbf{v}$, and D_N is a differentiate matrix (see section 2.1 of CANUTO et al. (2006) for details).

As we commented, equations (2.13) and (2.14) are usually solved in a $O(N^2)$ floating points operations. However, using the FFT, this calculation can be performed in $O(N \log N)$ floating point operations, as we mentioned in section 2.2.2.

2.4 Some Results about Functional Analysis

In this section, we present some definitions and important results about functional analysis that are used in the following chapters. The proofs of these results are omitted but properly referenced.

Definition 1. *Let $(X, \mathcal{A}, \mu)$ a measure space, $Y = \mathbb{R}$ or $\mathbb{C}$ and $p \in [1, \infty)$. The space $\mathcal{L}^p(X) = \mathcal{L}^p(X, \mathcal{A}, \mu, Y)$ is defined as*

$$\mathcal{L}^p(X) = \left\{ f : X \to Y | f \textit{ is measurable and } \int_X |f|^p d\mu < \infty \right\}.$$

Definition 2. *Let $Y = \mathbb{R}$ or $\mathbb{C}$. A function $f : X \to Y$ is essentially bounded if there exists some $r \in \mathbb{R}$ such that $|f(x)| \leq r$ almost everywhere in X.*

Definition 3. *Let $(X, \mathcal{A}, \mu)$ a measure space and $Y = \mathbb{R}$ or $\mathbb{C}$. The space $\mathcal{L}^\infty(X) = \mathcal{L}^\infty(X, \mathcal{A}, \mu, Y)$ is defined as*

$$\mathcal{L}^\infty(X) = \{ f : X \to Y | f \textit{ is measurable and essentially bounded} \}.$$

The $\mathcal{L}^p$ and $\mathcal{L}^\infty$ spaces defined are real (or complex) vector spaces, where

$$\|f\|_p = \Big(\int_X |f|^p d\mu \Big)^{1/p}, \quad f \in \mathcal{L}^p,$$

is a semi norm in $\mathcal{L}^p$, and

$$\|f\|_\infty = \inf \{ r \geq 0, |f| \leq r \text{ almost everywhere in } X \}, \quad f \in \mathcal{L}^\infty,$$

is a semi norm in $\mathcal{L}^\infty$.

Definition 4. *Let $(X, \mathcal{A}, \mu)$ a measure space, $Y = \mathbb{R}$ or $\mathbb{C}$ and $p \in [1, \infty)$. The space $L^p(X) = L^p(X, \mathcal{A}, \mu, Y)$ is defined as the quotient space with respect to the equivalence relation $f \sim g \Leftrightarrow f = g$ almost everywhere in X. That is,*

$$L^p(X) = \{[f], f \in L^p\},$$

where $[f] = \{g \in \mathcal{L}^p(X), g = f$ almost everywhere in $X\}$.

The $L^p(X)$ space is a Banach space with respect to the norm $\|[f]\|_p = \|f\|_p$, $p \in [1, \infty]$.

The continuous Fourier transform defined in section 2.2.1 of a function $f \in L^1(\mathbb{R})$ is well defined for all $k \in \mathbb{R}$, being a uniformly continuous and bounded function, such that $\|\hat{f}\|_\infty \leq \|f\|_1$.

Lemma 1 (Riemann-Lebesgue)**.** *Let $f : \mathbb{R} \to \mathbb{C}$ be an absolutely integrable function, such that f is piecewise continuous in each interval $[a, b] \subseteq \mathbb{R}$. Then, $\hat{f}(k) \to 0$ as $|k|$ approaches infinity.*

Proof. See KREYSZIG (1978). □

Definition 5 (Schwartz space). *The Schwartz space or space of rapidly decreasing functions, denoted by $\mathcal{S}(\mathbb{R})$, is the set of the functions $f : \mathbb{R} \to \mathbb{C}$ such that $f \in C^\infty(\mathbb{R})$ and*

$$\lim_{|x| \to +\infty} x^k D^\alpha f(x) = 0,$$

for all $k, \alpha \in \mathbb{N}$, where D^α denotes the αth derivative.

The inverse countinuous Fourier transform (2.2) is well defined for functions $f \in \mathcal{S}(\mathbb{R})$.

Definition 6 (Sequence spaces). *Let $p \in [1, \infty)$. The space $\ell^p = \ell^p(\mathbb{Z})$ is defined as*

$$\ell^p(\mathbb{Z}) = \left\{ \boldsymbol{u} = (u_j)_{j \in \mathbb{Z}}, \sum_{k=-\infty}^{\infty} |u_k|^p < \infty \right\},$$

with the norm $\|\boldsymbol{u}\|_p = \left(\sum_{k=-\infty}^{\infty} |u_k|^p \right)^{1/p}$. Similarly, the space $\ell^\infty = \ell^\infty(\mathbb{Z})$ is defined as

$$\ell^\infty(\mathbb{Z}) = \left\{ \boldsymbol{u} = (u_j)_{j \in \mathbb{Z}}, \sup_{k \in \mathbb{Z}} |u_k| < \infty \right\},$$

with norm $\|\boldsymbol{u}\|_\infty = \sup_{k \in \mathbb{Z}} |u_k|$.

The ℓ^p and ℓ^∞ are also Banach spaces. The formulation of the DFT and the inverse DFT given in section 2.2.2 are well defined when we consider the function $v \in \ell^2_N$, which is the set of functions defined at each mesh point $\{x_j\}$, N-periodic with respect to j (or 2π-periodic with respect to x), with the norm

$$\|v\| = \left(\Delta x \sum_{j=1}^{N} |v_j|^2 \right)^{1/2},$$

where Δx is the spacial step size used to discretize the the 2π- periodic interval.

Definition 7 (Sobolev spaces). *Let $s \in \mathbb{R}$. The Sobolev space $H^s_{per} = H^s_{per}([-\pi, \pi])$ is the set of all functions $f \in \mathcal{P}'$, which is the set of periodic distributions, such that*

$$\|f\|_s^2 = 2\pi \sum_{k=-\infty}^{\infty} (1 + |k|^2)^s |\hat{f}(k)|^2 < \infty.$$

In other words, a periodic distribution f is in H^s_{per} if and only if $\Big((1+|k|^2)^{\frac{s}{2}}|\hat{f}(k)|\Big)_{k\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$.

We denote by $\ell^2_s = \ell^2_s(\mathbb{Z})$ the space of sequences $\alpha = \alpha_k$, $k \in \mathbb{Z}$, such that

$$\|\alpha\|_{\ell^2_s} = \Big(2\pi \sum_{k=-\infty}^{\infty} (1+|k|^2)^s |\alpha_k|^2\Big)^{1/2}.$$

Therefore, $f \in H^s_{per}$ if and only if $\Big(\hat{f}(k)\Big)_{k\in\mathbb{Z}} \in \ell^2_s$. In this case, $\|f\|_s = \|\hat{f}\|_{\ell^2_s}$. It also can be shown that for $s \in \mathbb{R}$, H^s_{per} is a Hilbert space with respect to the inner product

$$(f,g)_s = 2\pi \sum_{k=-\infty}^{\infty} (1+|k|^2)^s \hat{f}(k)\overline{\hat{g}(k)}.$$

Observe that when $s = 0$, we have a Hilbert space that is isometrically isomorphic to $L^2([-\pi,\pi])$. The following results are all valid.

Proposition 1. *Let $s \geq r \in \mathbb{R}$. Then, $H^s_{per} \hookrightarrow H^r_{per}$ continuously, and $\|f\|_r \leq C\|f\|_s$ for all $f \in H^s_{per}$. In particular, if $s \geq 0$, then $H^s_{per} \subset L^2([-\pi,\pi])$. Moreover, the topological dual space, $(H^s_{per})'$, of H^s_{per} is isometrically isomorphic to H^{-s}_{per} for all $s \in \mathbb{R}$, and it holds that*

$$< f, g >_{H^{-s}_{per}\times H^s_{per}} = 2\pi \sum_{k=-\infty}^{\infty} \hat{f}(k)\hat{g}(k).$$

Proposition 2. *Let $m \in \mathbb{N}$. Then, $f \in H^m_{per}$ if and only if $\partial^j f = f^{(j)} \in L^2_{per}$, $j = 0, 1, \ldots, m$, where the derivatives are taking in the distribution sense (differentiation in $\mathcal{P}'$). Moreover, $\|f\|_m$ and*

$$\|\ |f|\ \|_m = \Big(\sum_{j=0} m\|\partial^j f^2\|_{L^2_{per}}\Big)^{1/2},$$

are equivalent, which means that there exist positive constants C_m and C'_m such that

$$C_m\|f\|^2_m \leq \|\ |f|\ \|^2_m \leq C'_m\|f\|^2_m, \quad f \in H^m_{per}.$$

Lemma 2 (Sobolev Lemma). *Let* $s > \frac{1}{2}$. *Then,* $H^s_{per} \hookrightarrow C_{per}$ *and* $\|f\|_\infty \leq \|\hat{f}\|_{\ell^1} \leq C\|f\|_s$ *for all* $f \in H^s_{per}$.

The proofs of the last two propositions and the Sobolev lemma can be found in KREYSZIG (1978).

2.5 Numerical Methods for Solving ODEs

Runge-Kutta (RK) methods along the linear multistep methods are the two most popular families of methods for solving numerically ordinary differential equations (ODEs). In this section, we give the basic and general concepts of local truncation error, consistency and order of accuracy of such methods, emphasizing what is valid for the RK methods.

In order to perform this presentation, we consider a nonlinear ODE system given by

$$\frac{dy}{dt} \equiv y' = f(t, y), \quad t > 0, \tag{2.15}$$

subject to initial conditions $y(0) = y_0$.

In general, y and f have m components each, $m \geq 1$. Here, to simplyfy the analysis, we consider $m = 1$, but the RK method described in the following can be readily generalized to ODE systems.

2.5.1 Runge-Kutta Methods

Runge–Kutta are one-step methods in which f in (2.15) is repeatedly evaluated within one mesh interval to obtain a higher order method. In general, an

s-stage Runge–Kutta method for the ODE (2.15) can be written in the form

$$Y_i = y_n + \Delta t \sum_{j=1}^{s} a_{ij} f(t_n + c_j \Delta t, Y_j), \quad 1 \leq i \leq s, \tag{2.16}$$

$$y_{n+1} = y_n + \Delta t \sum_{i=1}^{s} b_i f(t_n + c_i \Delta t, Y_i), \tag{2.17}$$

where each $Y_i,\ i = 1, \ldots, s$ is an intermediate approximation to the solution at time $t_n + c_i \Delta t$, which may be correct to a lower order of accuracy than the order for the solution y_{n+1} at the end of the step.

We shall always require that

$$1 = \sum_{j=1}^{s} b_j, \quad c_i = \sum_{j=1}^{s} a_{ij}, \quad i = 1, \ldots, s.$$

The coefficients of the RK methods are chosen in part to make error terms cancel in such a way that y_{n+1} is more accurate. As for multistep methods, the local truncation error d_n is defined as the amount by which the exact solution fails to satisfy the numerical formula per step (2.17), divided by Δt. The order of the method is then p if $d_n = O(\Delta t^p)$. A method whose order is at least 1 is said to be consistent (see ASCHER (2008) for details).

The RK method is explicit if $a_{ij} = 0$, for all $i \leq j$; we can cite as examples, the forward Euler method and the classical fourth order Runge-Kutta method (RK4). In general, the order p of an explicit RK method satisfies $p \leq s$, with $p = s$ for both forward Euler and the RK4 methods. There are no such methods with $p = s > 4$.

If the RK method is not explicit, then it is implicit. The backward Euler method can be cited as an example of an implicit RK method. More details about the order of convergence for this type of RK method can be found in ASCHER (2008).

Remark 1. *When $s = 4$ in (2.16)-(2.17), we have the fourth order Runge Kutta method, or RK4 as already mentioned. A simpler version of these equations in this case is given by*

$$
\begin{aligned}
&y_0 = y(0) \quad \textit{(initial data)},\\
&k_1 = \Delta t f(t_i, y_i),\\
&k_2 = \Delta t f\left(t_i + \tfrac{\Delta t}{2}, y_i + \tfrac{1}{2}k_1\right),\\
&k_3 = \Delta t f\left(t_i + \tfrac{\Delta t}{2}, y_i + \tfrac{1}{2}k_2\right),\\
&k_4 = \Delta t f(t_{i+1}, y_i + k_3),\\
&y_{i+1} = y_i + \tfrac{1}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right).
\end{aligned}
$$

for each $i = 0, 1, \ldots, M-1$, where $M > 0$ denotes the number of steps in the time interval. The intermediate steps are represented here by k_1, k_2, k_3, k_4 to eliminate the need for successive nesting in the second variable of $f(t, y)$. More details about this can be found in BURDEN; FAIRES (2010).

2.5.2 Convergence and Stability

The initial approach for solving an ODE system is to ask what is required for the ODE problem to be well-posed and then to require that the numerical method behave similarly. In this case, behave similarly means to choose a stable numerical method; the stability of a numerical method behaves, in a similarly way, as the well posedness of the initial value problem (IVP).

In fact, for ODEs, the requirement for well-posedness can be easily stated in a more general way, requiring only Lipschitz continuity of f in terms of y. Here is the fundamental theorem for initial value ODEs.

Theorem 1. *Let $f(t, y)$ be continuous for all (t, y) in a region $\mathcal{D} = \{0 \leq t \leq b, |y| < \infty\}$. Moreover, assume Lipschitz continuity of f in y: there exists a constant L such that for all (t, y) and $(t, \hat{y})$ in $\mathcal{D}$, $|f(t, y) - f(t, \hat{y})| \leq L|y - \hat{y}|$. Then,*

- *For any initial value vector $y_0 \in \mathbb{R}^n$ there exists a unique solution $y(t)$ for the IVP throughout the interval $[0, b]$. This solution is differentiable.*

- *The solution y depends continuously on the initial data: if $\hat{y}$ also satisfies the ODE (but not the same initial values), then*

$$|y(t) - \hat{y}(t)| \leq e^{Lt}|y(0) - \hat{y}(0)|,$$

- *If $\hat{y}$ satisfies, more generally, a perturbed ODE*

$$\hat{y}' = f(t, \hat{y}) + r(t, \hat{y}),$$

where r is bounded on $\mathcal{D}$, $|r| \leq M$, then

$$|y(t) - \hat{y}(t)| \leq e^{Lt}|y(0) - \hat{y}(0)| + \frac{M}{L}(e^{Lt} - 1).$$

Proof. See ASCHER (2008). □

Thus we have solution existence, uniqueness, and continuous dependence on the data, or in other words, a well-posed problem, provided that the conditions of the theorem hold.

Considering next the numerical method approximating the ODE problem (2.15), let y_π be a mesh function which takes on the value y_π at each t_n, i.e., $y_\pi(t_n) = y_n$, $n = 0, 1, \ldots, N$. The function $y_\pi(t)$ can be seen as a function defined for all t, but all that matters is what it does at the mesh points.

Then, the numerical method is given by $\mathcal{N}_\pi y_\pi(t_n) = 0$, where $\mathcal{N}_\pi$ may be taken as a finite difference operator. Under the conditions of the Theorem 1, we define the numerical method to be 0-*stable* if there are positive constants k_0 and $\mathcal{K}$ such that for any mesh functions x_π and z_π with $\Delta t \leq k_0$

$$|x_\pi - z_\pi| \leq \mathcal{K}\left\{|x_0 - z_0| + \max_{1\leq j\leq N}|\mathcal{N}_\pi x_\pi(t_j) - \mathcal{N}_\pi z_\pi(t_j)|\right\}, \ 1 \leq j \leq M. \tag{2.18}$$

What this bound says in effect is that the difference operator $\mathcal{N}_\pi$ is invertible and that its inverse is bounded by $\mathcal{K}$. The definition (2.18) lead to proof the following result.

Theorem 2.

$$\textit{Consistency} + \textit{0-Stability} \;\Rightarrow\; \textit{Convergence}$$

Indeed, if the method is consistent of order p and 0*-stable, then it is convergent of order p, and*

$$|y_n - y(t_n)| \equiv |e_n| \leq K \; \underset{j}{max}|d_j| = O(k^p).$$

Proof. See ASCHER (2008). □

It is important to distinguish between the stability concepts arising in PDEs and those arising in ODEs. They are related but are not quite the same. More details about this can be found in ASCHER (2008).

2.5.3 Absolute Stability

In the following, we talk about the *absolute stability* of numerical methods for solving ODEs. To perform this, we consider the test equation $y' = \lambda y,\ t > 0$, where λ is a constant, complex scaler standing. The solution of this test equation is $y(t) = e^{\lambda t}y(0)$, and satisfies that $|y(t)| \leq |y(0)| \Leftrightarrow \mathcal{R}e\ \lambda > 0$.

Correspondingly, for a numerical method we define the region of *absolute stability* as that region of the complex z-plane containing the origin where $|y_{n+1}| \leq |y_n|$, $n = 0, 1, \ldots$, when applying the method for the test equation, with $z = k\lambda$ from within this region. Note that 0-stability for the test equation corresponds to having the origin $z = 0$ belong to the absolute stability region.

For a RK method in the general form (2.16)-(2.17) applied to the test equation, we have

$$Y_i = y_n + z\sum_{j=1}^{s} a_{ij}Y_j, \quad y_{n+1} = y_n + z\sum_{j=1}^{s} b_jY_j.$$

Eliminating the internal stages $Y_1, \ldots, Y_s$ and substituting in the expression for y_{n+1}, we obtain

$$y_{n+1} = R(z)y_n, \quad R(z) = 1 + z\mathbf{b}^T(I - zA)^{-1}\mathbf{1}, \tag{2.19}$$

where the term $\mathbf{b}^T(I - zA)^{-1}\mathbf{1}$ is defined similarly to the equation (2.13) in chapter 2 of ASCHER (2008).

The stability regions for explicit Runge–Kutta methods of the first few orders are showed in Figure 2.2 .

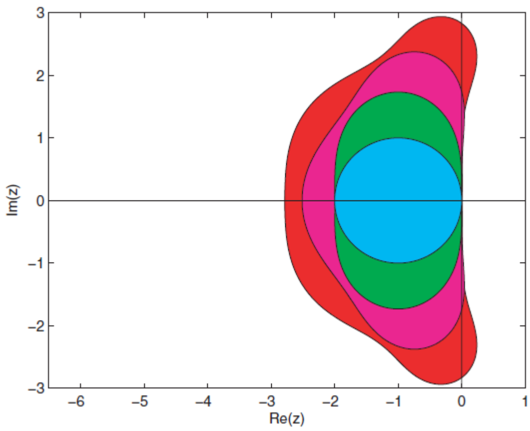

Figure 2.2: Stability regions for p-stage explicit RK methods of order $p = 1, 2, 3, 4$. Source: ASCHER (2008).

In Figure 2.2, the blue region represents the stability region for 1-stage RK method; the green region for the 2-stage RK method; the pink and the red regions represent the stability regions for 3,4-stage RK method, respectively. It is important to note that whereas the regions for the Runge–Kutta methods of order 1 and 2 have no intersection with the imaginary axis, those for the higher order methods do. The observation in Figure 2.2 for the classical RK4 regarding the intersection of its absolute stability region with the imaginary axis is a major reason for the popularity of this method in time discretizations of hyperbolic PDEs (see ASCHER (2008)).

2.6 Important inequalities and notations

Cauchy-Schwartz inequality in $L^2(\Omega)$

Let $f : \Omega \longrightarrow \mathbb{R}$ and $g : \Omega \longrightarrow \mathbb{R}$ be two square integrable functions. Then,

$$|(f,g)|_{L^2} = \left|\int_\Omega f(x)g(x)\,dx\right| \le \left[\int_\Omega |f(x)|^2\,dx\right]^{\frac{1}{2}} \left[\int_\Omega |fg(x)|^2\,dx\right]^{\frac{1}{2}} = \|f\|_{L^2}\|g\|_{L^2}.$$

Hölder inequality

Let $p_i \ge 1$, $i = 1, 2, ..., m$, such that

$$\sum_{i=1}^{m} \frac{1}{p_i} = 1.$$

If $f_i \in L^{p_i}(\Omega)$ for $i = 1, 2, ..., m$, then it holds that $\prod_{i=1}^{m} f_i \in L^1(\Omega)$ and

$$\int_\Omega \left|\prod_{i=1}^{m} f_i(x)\right|\,dx \le \prod_{i=1}^{m}\left(\int_\Omega |f_i(x)|^{p_i}\,dx\right)^{\frac{1}{p_i}}.$$

Young inequality

Let $a, b \ge 0$ and $p, q \ge 0$ constants such that $\frac{1}{p} + \frac{1}{q} = 1$. Then, it holds that

$$ab \le \frac{a^p}{p} + \frac{b^q}{q}$$

.

Continuous Gronwall inequality

Let f, g and h be positive functions satisfying

$$f(t) + h(t) \leq g(t) + c\int_a^t f(s)ds, \quad \forall t \in [0,T].$$

Then, it holds that

$$f(t) + h(t) \leq e^{c(t-a)}g(t).$$

Discrete Gronwall inequality

Let k_n be a sequence of non negative real numbers. Consider a sequence $\phi_n \geq 0$ such that $\phi_0 \leq g_0$,

$$\phi_n \leq g_0 + \sum_{s=0}^{n-1} p_s + \sum_{s=0}^{n-1} k_s\phi_s, \; n \geq 1,$$

with $g_0 \geq 0$ and $k_s \geq 0$. Then, for all $n \geq 1$, it holds that

$$\phi_n \leq \left(g_0 + \sum_{s=0}^{n-1} p_s\right) exp\left\{\sum_{s=0}^{n-1} k_s\right\}.$$

Notation

The standard norm in $L^p(\mathbb{R})$ will be written $|\cdot|_p$ for $1 \leq p \leq \infty$. If $f \in H^s = H^s(\mathbb{R})$, where $s \geq 0$, the Sobolev class of L^2-functions whose first s derivatives are also belong in L^2, then its norm is written $\|f\|_s$. If s is not an integer, the notation is extended via the Fourier transform in the usual way; see BONA; CHEN; SAUT (2002) for details.

3 BOUSSINESQ SYSTEMS

In this chapter, we give an overview of the principal aspects of the Boussinesq systems (1.12), specifically its deduction from the Euler equations and well posedness results.

3.1 Deduction from Euler Equations

We consider the 2D Euler equations gave in chapter 1 by equations (1.1)-(1.9), where Ψ satisfies the compatibility condition

$$\frac{\partial \Psi}{\partial y}(x,y) = -\int_{h_0}^{y} \frac{\partial^2 \Psi}{\partial x^2}(x,y^{'})dy^{'} \quad \text{in} \quad \Omega_0,$$

which follows by assuming that (1.3), (1.4) and (1.7) hold at $t = 0$.

The first step to obtain (1.12) from the Euler equations is to remove, parcial or totally, the dimensionalization of (1.1)-(1.9), scaling these equations in an appropriate way. Since to scale a problem is related with the nature of the problem, and we are interested in the regime of long surface waves of small amplitude, we consider the variables ε and σ such that,

$$\varepsilon := \frac{A}{h_0} \ll 1, \quad \sigma := \frac{h_0}{\lambda} \ll 1, \tag{3.1}$$

where A and λ represent, respectively, a typical amplitude and wavelength of the waves.

Following PEREGRINE (1972), we make the change of variables

$$\begin{gathered} x^* = \tfrac{\sigma}{h_0}x, \quad y^* = \tfrac{1}{h_0}y, \quad t^* = \tfrac{\sigma g}{c_0}t, \quad \eta^* = \tfrac{1}{\varepsilon h_0}\eta, \\ u^* = \tfrac{1}{\varepsilon c_0}u, \quad v^* = \tfrac{1}{\varepsilon\sigma}\tfrac{v}{c_0}, \quad p^* = \tfrac{1}{\rho c_0^2}p, \end{gathered}$$

where $c_0 = \sqrt{gh_0}$.

With respect to this new variables, we obtain that

$$u = \varepsilon c_0 u^* \quad \Rightarrow \quad \begin{array}{l} u_t = \varepsilon\sigma g u^*_{t^*}, \\ u_x = \varepsilon c_0 u^*_{x^*}\frac{\sigma}{h_0}, \\ u_y = \varepsilon c_0 u^*_{y^*}\frac{1}{h_0}, \end{array} \tag{3.2}$$

$$p = \rho c_0^2 p^* \quad \Rightarrow \quad \begin{array}{l} p_x = \rho\sigma g p^*_{x^*}, \\ p_y = \rho g p^*_{y^*}\frac{1}{h_0}, \end{array} \tag{3.3}$$

$$v = \varepsilon\sigma c_0 v^* \quad \Rightarrow \quad \begin{array}{l} v_x = \varepsilon\sigma^2\frac{c_0}{h_0}v^*_{x^*}, \\ v_y = \varepsilon\sigma c_0 v^*_{y^*}\frac{1}{h_0}. \end{array} \tag{3.4}$$

Then, using (3.2), (3.3) and (3.4), we can rewrite the system (1.1)-(1.9) as

$$\varepsilon u^*_{t^*} + \varepsilon^2 u^* u^*_{x^*} + \varepsilon^2 v^* u^*_{y^*} + p^*_{x^*} = 0, \tag{3.5}$$

$$\varepsilon\sigma^2 v^*_{t^*} + \varepsilon^2\sigma^2 u^* v^*_{x^*} + \varepsilon^2\sigma^2 v^* v^*_{y^*} + p^*_{y^*} = -1, \tag{3.6}$$

$$u^*_{x^*} + v^*_{y^*} = 0, \tag{3.7}$$

$$u^*_{y^*} - \sigma^2 v^*_{x^*} = 0, \tag{3.8}$$

with boundary and initial conditions given by

$$\eta^*_{t^*} + \varepsilon u^* \eta^*_{x^*} = v^* \ \text{ if } \ y^* = \varepsilon\eta^*(x^*, t^*), \tag{3.9}$$

$$p^* = 0 \ \text{ if } \ y^* = \varepsilon\eta^*(x^*, t^*), \tag{3.10}$$

$$v^* = 0\ , \ \text{ if } \ y^* = -1, \tag{3.11}$$

$$\eta^*(x^*, 0) = \Phi^*(x^*), \tag{3.12}$$

$$u^*(x^*, y^*, 0) = \Psi^*(x^*, y^*), \tag{3.13}$$

where the initial conditions are defined, in terms of Φ and Ψ, by

$$\Phi^*(x^*) := \frac{1}{h_0\varepsilon}\Phi(\frac{x^* h_0}{\sigma}), \quad \Psi^*(x^*, y^*) := \frac{1}{\varepsilon c_0}\Psi(\frac{h_0}{\sigma}x^*, h_0 y^*). \tag{3.14}$$

We can solve the system (3.7)-(3.8)-(3.11), and considering, for example, $U^* = U^*(x^*, t^*) := u^*(x^*, 0, t^*)$, we can represent its solution by

$$u^*(x^*, y^*, t^*) = U^* - \sigma^2\Big(y^* + \frac{1}{2}(y^*)^2\Big)U^*_{x^*x^*} + \mathrm{O}(\sigma^4), \quad (3.15)$$

$$v^*(x^*, y^*, t^*) = -(y^* + 1)U^*_{x^*} + \frac{\sigma^2}{6}(3(y^*)^2 + (y^*)^3 - 2)U^*_{x^*x^*x^*} + \mathrm{O}(\sigma^4). \quad (3.16)$$

On the other hand, using (3.6) along with (3.15), (3.16) and (3.10), we obtain

$$p^*(x^*, y^*, t^*) = -y^* + \varepsilon\eta^* + \varepsilon\sigma^2 y^*\Big(1 + \frac{1}{2}y^*\Big)U^*_{x^*t^*} + \mathrm{O}(\varepsilon\sigma^4, \varepsilon^2\sigma^2). \quad (3.17)$$

Using (3.15), (3.16) and (3.17) in (3.5), we get

$$U^*_{t^*} + \eta^*_{x^*} + \varepsilon U^* U^*_{x^*} = \mathrm{O}(\sigma^4, \varepsilon\sigma^2). \quad (3.18)$$

Another equation coupling U^* and η^* is obtained by evaluating (3.15) and (3.16) at $y^* = \varepsilon\eta^*$ and substituting in em (3.9). This gives us

$$\eta^*_{t^*} + U^*_{x^*} + \varepsilon(\eta^* U^*_{x^*}) + \frac{\sigma^2}{3}U^*_{x^*x^*x^*} = \mathrm{O}(\varepsilon\sigma^2, \sigma^4). \quad (3.19)$$

If we assume that the dispersive and nonlinear terms $U^*_{x^*x^*x^*}$ and $U^*U^*_{x^*}$, respectively, in the systems (3.18) and (3.19) are of equal importance, that is, the Stokes number $S = \dfrac{\varepsilon}{\sigma^2} = \mathrm{O}(1)$, and put for definiteness $\varepsilon = \sigma^2$, we obtain the Boussinesq type system

$$\begin{aligned} \eta^*_{t^*} + U^*_{x^*} + \varepsilon(\eta^* U^*)_{x^*} + \tfrac{\varepsilon}{3}U^*_{x^*x^*x^*} &= 0, \\ U^*_{t^*} + \eta^*_{x^*} + \varepsilon U^* U^*_{x^*} &= 0. \end{aligned} \quad (3.20)$$

Following BONA; CHEN; SAUT (2002), we choose as velocity variable the horizontal velocity of the fluid $u^*_\theta(x^*, t^*)$ at the height $y^* = -1 + \theta(1 + \varepsilon\eta^*)$, for some

$\theta \in [0,1]$. That is equivalent to set $u_\theta^*(x^*,t^*) = u^*(x^*, -1+\theta(1+\varepsilon\eta^*), t^*)$. Taylor expansions and the use of (3.15) with $\sigma^2 = \varepsilon$, let us to obtain

$$u_\theta^* = U^* + \frac{1}{2}\varepsilon(1-\theta^2)U^*_{x^*x^*} + \mathrm{O}(\varepsilon^2),$$

which we may invert, using e.g., the Fourier transform, to obtain

$$U^* = u_\theta^* - \frac{1}{2}\varepsilon(1-\theta^2)u^*_{\theta,x^*x^*} + \mathrm{O}(\varepsilon^2). \tag{3.21}$$

Substituting this expression into the system (3.20), we obtain the Boussinesq system

$$\begin{aligned} \eta^*_{t^*} + u^*_{\theta,x^*} + \varepsilon(\eta^* u^*_\theta)_{x^*} + \tfrac{\varepsilon}{2}\left(\theta^2 - \tfrac{1}{3}\right)u^*_{\theta,x^*x^*x^*} &= 0, \\ u^*_{\theta,t^*} + \eta^*_{x^*} + \varepsilon u^*_\theta u^*_{\theta,x^*} - \tfrac{\varepsilon}{2}(1-\theta^2)u^*_{\theta,x^*x^*t^*} &= 0, \end{aligned} \tag{3.22}$$

valid for $x^* \in \mathbb{R}$, $t^* \geq 0$, with initial conditions given by

$$\eta^*(x^*,0) = \eta_0^*(x^*) := \Phi^*(x^*), \tag{3.23}$$

and

$$u_\theta^*(x^*,0) = u^*_{\theta,0}(x^*) := \Psi^*(x^*, -1+\theta(1+\varepsilon\eta_0^*(x^*))), \tag{3.24}$$

with Φ^* and Ψ^* given in (3.14).

Observing that, since u_θ^* and η^* are determined by (3.22)-(3.23)-(3.24), we can compute the functions U^*, u^*, v^* and p^* from the relations given by (3.15), (3.16), (3.17) e (3.21).

Following BONA; CHEN; SAUT (2002) and BENJAMIN; BONA; MAHONY (1972), we observe that (3.22) gives us that

$$\begin{aligned} \eta^*_{t^*} + u^*_{\theta,x^*} &= \mathrm{O}(\varepsilon), \\ u^*_{\theta,t^*} + \eta^*_{x^*} &= \mathrm{O}(\varepsilon), \end{aligned}$$

which allows us to conclude that the third order derivatives of u_θ^* in (3.22) may be expressed in terms of the third order derivatives of η^* with an error of $\mathrm{O}(\varepsilon)$, namely

$u_{\theta,x^*x^*x^*} = -\eta^*_{x^*x^*t^*} + \mathrm{O}(\varepsilon)$. More generally, using a real parameter ν, we can write

$$u_{\theta,x^*x^*x^*} = \nu u_{\theta,x^*x^*x^*} - (1-\nu)\eta^*_{x^*x^*t^*} + \mathrm{O}(\varepsilon).$$

Using similar arguments, we can write $u_{\theta,x^*x^*t^*}$ depending on another real parameter, namely μ, and using such expressions in (3.22), we obtain a family of Boussinesq systems in nondimensional, scaled variables given by

$$\begin{aligned} \eta^*_{t^*} + u^*_{\theta,x^*} + \varepsilon(\eta^* u^*_\theta)_{x^*} + \varepsilon(a u_{\theta,x^*x^*x^*} - b\eta^*_{x^*x^*t^*}) = 0, \\ u^*_{\theta,t^*} + \eta^*_{x^*} + \varepsilon u^*_\theta u^*_{\theta,x^*} + \varepsilon(c\eta^*_{x^*x^*x^*} - d u_{\theta,x^*x^*t^*}) = 0, \end{aligned} \tag{3.25}$$

where the constants a, b, c, d are given in (1.13), with $\theta \in [0,1]$ and $\nu, \mu \in \mathbb{R}$. The system (3.25) is supplemented with the initial conditions (3.23) e (3.24).

The next step is to eliminate the dependence of the scale variable ε of the family (3.25), whenever this scale variable plays no essencial rule in the analytical or numerical analysis. Considering the change of variables given by

$$\tilde{u} = \varepsilon u^*_\theta, \quad \tilde{\eta} = \varepsilon \eta^*, \quad \tilde{x} = \varepsilon^{-1/2} x^*, \quad \tilde{t} = \varepsilon^{-1/2} t^*,$$

we obtain the system

$$\begin{aligned} \tilde{\eta}_{\tilde{t}} + \tilde{u}_{\tilde{x}} + (\tilde{\eta}\tilde{u})_{\tilde{x}} + a\tilde{u}_{\tilde{x}\tilde{x}\tilde{x}} - b\tilde{\eta}_{\tilde{x}\tilde{x}\tilde{t}} = 0, \\ \tilde{u}_{\tilde{t}} + \tilde{\eta}_{\tilde{x}} + \tilde{u}\tilde{u}_{\tilde{x}} + c\tilde{\eta}_{\tilde{x}\tilde{x}\tilde{x}} - d\tilde{u}_{\tilde{x}\tilde{x}\tilde{t}} = 0, \end{aligned} \tag{3.26}$$

for $\tilde{x} \in \mathbb{R}$ and $\tilde{t} \geq 0$, with initial conditions

$$\tilde{\eta}(\tilde{x}, 0) = \tilde{\eta}_0(\tilde{x}), \qquad \tilde{u}(\tilde{x}, 0) = \tilde{u}_0(\tilde{x}),$$

obtained by (3.23) and (3.24) using the last mentioned change of variables.

Since (3.26) is intended to model waves of small amplitude and large wave-length, we have that $\tilde{\eta}_0 = \mathrm{O}(\varepsilon)$, as well as $\tilde{u}_0$. This means that the system (3.26) should be integrated with small initial data. It is also worth noticing that, during the approximation process which leads us to the system (3.26), one-way propagation assumptions were nowhere made. Therefore, these systems can be used to study the two-way propagation of long surface waves of small amplitude.

Finally, if we omit the tilde in (3.26) in order to simplify the notation, we get the Boussinesq system represented by

$$\begin{aligned} \eta_t + u_x + (u\eta)_x + au_{xxx} - b\eta_{xxt} &= 0, \\ u_t + \eta_x + uu_x + c\eta_{xxx} - du_{xxt} &= 0, \end{aligned}$$

with the constants a, b, c, d given in (1.13).

3.2 Examples of Boussinesq Systems

We have several possibilities for the parameters a, b, c, d in (1.13), which give rise to different but equivalent systems, in the sense that all of them model surface waves propagation. Some particular choices of these parameters a, b, c, d, have often appeared in the literature. These particular systems have interesting characteristics (see KAMPANIS; DOUGALIS; EKATERINARIS (2008) for more information), such that

(i) *Have favorable mathematical properties.* Some of theses systems are linearly ill-posed and should be excluded from further study as useful model equations. Among the linearly well-posed ones, we should consider systems whose Cauchy problem is at least locally (nonlinearly) well-posed with long enough temporal interval of existence solutions. Moreover, existence of solutions to initial-boundary-value problem (IBVP) is an important requirement. It is also worth noticing that these systems in their scaled form should be rigorously justified to be good approximations of the Euler equations as $\varepsilon \to 0$.

(ii) *Have solitary waves solutions.* Since Boussinesq systems are approximations to Euler equation, it is expected that they have *solitary wave* solutions. Therefore, attention must be paid to systems that have solitary waves, whose uniqueness, stability and other properties should be also studied.

(iii) *Can be easily solved numerically with high accuracy.* The scarcity of closed form analytical solutions and the need to study, for example, the stability and the interactions of the solitary waves of some of theses systems, as well as the characteristics of the long-time evolution of their solutions make it imperative that the systems should be solved numerically by fully discrete methods.

A forth, and perhaps the most important, criterion for selecting a 'good' system is, of course, the favorable comparison of its solutions with experimental data of propagation of surface waves. This issue of modeling is studied in detail in BONA; CHEN; SAUT (2002) and BONA; CHEN (1998).

In sequel, we list some examples of particular Boussinesq systems of the form (1.12), whose the initial-value problem (IVP) for all these systems have be shown to be at least nonlinearly well-posed locally in time (see BONA; CHEN; SAUT (2002)).

(i) 'Classical' Boussinesq system ($\theta^2 = \frac{1}{3}$, λ arbitrary, $\mu = 0$)

$$\begin{aligned} \eta_t + u_x + (u\eta)_x &= 0, \\ u_t + \eta_x + uu_x - \tfrac{1}{3}u_{xxt} &= 0, \end{aligned}$$

which the corresponding IVP is globally well-posed as can be seen in AMICK (1984) and SCHONBEK (1981).

(ii) BBM-BBM system ($\nu = \mu = 0$, $\theta^2 = \frac{2}{3}$, i.e., $a = c = 0$, $b = d = \frac{1}{6}$)

$$\begin{aligned} \eta_t + u_x + (u\eta)_x - \tfrac{1}{6}\eta_{xxt} &= 0, \\ u_t + \eta_x + uu_x - \tfrac{1}{6}u_{xxt} &= 0, \end{aligned}$$

which the corresponding IVP is locally well-posed as can be seen in BONA; CHEN; SAUT (2004) and BONA; CHEN (1998).

(iii) Bona-Smith system ($\nu = 0$, $\mu = \frac{4-6\theta^2}{3(1-\theta^2)}$, i.e., $a = 0$, $b = d = \frac{3\theta^2-1}{6}$, $c = \frac{2-3\theta^2}{3}$, with $\frac{2}{3} < \theta^2 < 1$)

For these parameters, the IVP is globally well-posed as can be seen in BONA; SMITH (1976). The limiting form of this system as $\theta \to 1$, corresponding to $a = 0$, $b = d = \frac{1}{3}$ and $c = -\frac{1}{3}$, which it was studied by Bona and Smith in BONA; SMITH (1976), is given by

$$\begin{aligned} \eta_t + u_x + (u\eta)_x - \tfrac{3\theta^2-1}{6}\eta_{xxx} &= 0, \\ u_t + \eta_x + uu_x + \tfrac{2-3\theta^2}{3}\eta_{xxx} - \tfrac{3\theta^2-1}{6}u_{xxt} &= 0, \end{aligned}$$

(iv) KdV-KdV system ($\nu = \mu = 1$, $\theta^2 = \frac{2}{3}$, i.e., $a = c = \frac{1}{6}$ e $b = d = 0$)

$$\begin{aligned} \eta_t + u_x + (u\eta)_x - \tfrac{1}{6}u_{xxx} &= 0, \\ u_t + \eta_x + uu_x + \tfrac{1}{6}\eta_{xxx} &= 0, \end{aligned}$$

which the corresponding IVP is locally well-posed as can be seen in BONA; CHEN; SAUT (2002) and BONA; DOUGALIS; MITSOTAKIS (2007).

3.3 Results about Well-Posedness of Boussinesq Systems

In this section we briefly present the most relevant results concerning the existence and uniqueness of solutions of the Boussinesq system that were presented in BONA; CHEN; SAUT (2002) AND BONA; CHEN; SAUT (2004). We divide the section in two sub sections, containing the results for the linear and nonlinear families.

3.3.1 Linear Models

The initial-value problem for the linear model obtained from the original system in section 3.1 by ignoring the nonlinear terms $(u\eta)_x$ and uu_x was analyzed in BONA; CHEN; SAUT (2002). The linear family of Boussinesq systems is given by

$$\begin{aligned} \eta_t + u_x + au_{xxx} - b\eta_{xxt} = 0, \\ u_t + \eta_x + c\eta_{xxx} - du_{xxt} = 0, \end{aligned} \tag{3.27}$$

where a, b, c, d are defined in (1.13).

The system (3.27) is straightforwardly understood using Fourier analysis, in the sense that this analysis gives us information about the well-posedness of the initial value problem.

Consider $\eta(0, x) = \eta_0(x)$ and $u(0, x) = u_0(x)$ for $x \in \mathbb{R}$ the initial conditions for system (3.27). Taking the Fourier transform with respect to x, the system (3.27) may be written in the form

$$\begin{pmatrix} \hat{\eta}(k,t) \\ \hat{u}(k,t) \end{pmatrix} = e^{-ik\mathcal{A}(k)t} \begin{pmatrix} \hat{\eta}_0(k) \\ \hat{u}_0(k) \end{pmatrix}, \tag{3.28}$$

where

$$\mathcal{A}(k) = \begin{pmatrix} 0 & w_1(k) \\ w_2(k) & 0 \end{pmatrix},$$

and

$$w_1(k) = \frac{1 - ak^2}{1 + bk^2}, \quad w_2(k) = \frac{1 - ck^2}{1 + dk^2}. \tag{3.29}$$

Using standard results from Fourier theory, in BONA; CHEN; SAUT (2002) it is shown that the well-posedeness of system (3.27) is guaranteed by the following result:

Proposition 3. *For a, b, c, d satisfying (1.13), the matrix $e^{-ik\mathcal{A}(k)t}$ is bounded on finite intervals of wavenumbers k if and only if one of the following sets of conditions* *hold:*

(C1) $b \geq 0,\ d \geq 0,\ a \leq 0,\ c \leq 0$*;*

(C2) $b \geq 0,\ d \geq 0,\ a = c > 0$*;*

(C3) $b = d < 0,\ a = c > 0$*.*

After straightforward computations we conclude that in all three cases (C1), (C2) and (C3), $w_1(k)w_2(k) \geq 0$ is true and then

$$e^{-ikA(k)t} = \begin{pmatrix} \cos(k\sigma(k)t) & -i\sin(k\sigma(k)t)\frac{w_1(k)}{\sigma} \\ -i\sin(k\sigma(k)t)\frac{w_2(k)}{\sigma} & \cos(k\sigma(k)t) \end{pmatrix}, \tag{3.30}$$

where $\sigma(k) = \sqrt{w_1(k)w_2(k)}$.

A consequence of (3.28) and (3.30) is that for any value of the index s, a solution of (3.27) satisfies

$$\|\eta\|_s^2 + \|\mathcal{H}u\|_s^2 = \|\eta_0\|_s^2 + \|\mathcal{H}u_0\|_s^2,$$

where $\mathcal{H}$ represents the Fourier multiplier operator defined by

$$\widehat{\mathcal{H}g}(k) = \left(\frac{w_1(k)}{w_2(k)}\right)^{\frac{1}{2}} \hat{g}(k).$$

Let the order of the operator $\mathcal{H}$ be the integer ℓ such that

$$\left(\frac{w_1(k)}{w_2(k)}\right)^{\frac{1}{2}} \sim C|k|^{\ell},$$

as k approaches infinity. Then $\mathcal{H}$ is a bijective bounded linear operator from H^s to $H^{s-\ell}$ for any index s.

As a consequence the following result about well-posedness for the initial-value problem for (3.27) follows (BONA; CHEN; SAUT, 2002, Theorem 3.2).

Theorem 3. *Let a, b, c, d satisfy one of the conditions (C1)-(C3) in Proposition 3. Let $m_1 = \max(0, -\ell)$, $m_2 = \max(0, \ell)$. Then, the corresponding linear initial-value problem (3.27) is well posed in $H^{s+m_1} \times H^{s+m_2}$, for any $s \geq 0$.*

Taking $s = 0$, the Theorem 3 implies that,

Order of $\mathcal{H}$	Well-Posed in
2	$H^0 \times H^2$
1	$H^0 \times H^1$
0	$H^0 \times H^0$
-1	$H^1 \times H^0$
-2	$H^2 \times H^0$

Remark 2. *The systems with $\mathcal{H}$ having order -2 are not admissible as models of the underlying physical situation. In this case, $a = d = 0$ and $b \neq 0$, $c \neq 0$, which is incompatible with (1.13) and any conditions in the Proposition 3.*

3.3.2 Nonlinear Models

The analysis of the nonlinear Boussinesq system was carried out in BONA; CHEN; SAUT (2004). There were studied the local and global well-posedness for the most relevant classes of problems arising from

$$\begin{aligned} \eta_t + u_x + (\eta u)_x + a u_{xxx} - b\eta_{xxt} = 0, \\ u_t + \eta_x + u u_x + c\eta_{xxx} - d u_{xxt} = 0, \end{aligned} \tag{3.31}$$

with the parameters a, b, c, d as in (1.13). More specifically, the authors studied the cases where conditions (C1) or (C2) are satisfied. The well-posedenss for this type of systems also was largely studied in FAWCETT (1992). Next, we present some of the results from BONA; CHEN; SAUT (2004) that are most important for this work.

Theorem 4. *(BONA; CHEN; SAUT, 2004, Theorems 3.12, 4.2 and 4.3) The systems in* (3.31) *with a, b, c, d satisfying* (1.13) *and (C1) or (C2) are all locally well posed in the corresponding Sobolev classes* $H^{s+m_1} \times H^{s+m_2}$*, for suitable values of* $s \geq 0$*, where* m_1, m_2 *are taken as in Theorem 3. Moreover, if* $b = d > 0$*,* $a \leq 0$*,* $c < 0$ *and* $s \geq 1$ *then for initial conditions satisfying*

$$\inf_{x \in \mathbb{R}} \{1 + \eta_0(x)\} > 0, \tag{3.32}$$

$$\left| \int (c\eta_{0,x}^2 + a u_{0,x}^2 - \eta_0^2 - u_0^2 - u_0^2 \eta_0) dx \right| < 2|c|^{1/2}, \tag{3.33}$$

the corresponding solutions are global (i.e. the local solutions can be extended for any $t \geq 0$*).*

In the following, we give the specific results known about well-posedness for various range of parameters of the system (3.31).

(i) Purely BBM-type Boussinesq system: $a = c = 0, b, d > 0$.

Theorem 5. *Let* $s \geq 0$ *be given. For any* $(\eta_0, u_0) \in H^s(\mathbb{R})^2$*, there is a* $T > 0$ *and a unique solution* (η, u) *of (3.31) that satisfies* $(\eta, u) \in C(0, T; H^s(\mathbb{R})^2)$*. Additionally, when* $s > \frac{1}{2}$*, then* $(\partial_t^k \eta, \partial_t^k u) \in C(0, T; H^{s+1}(\mathbb{R})^2)$ *for* $k = 1, 2, \ldots$*. Moreover, the correspondence* $(\eta_0, u_0) \mapsto (\eta, u)$ *is locally Lipschitz.*

(ii) Weakly dispersive systems: $b > 0, d > 0$.

- Case I: $\mathcal{H}$ has order 0, i.e., $a < 0, b > 0, c < 0, d > 0$ (the "generic"case) or $a = c > 0, b > 0, d > 0$.

  Theorem 6. *Let* $s \geq 0$ *and* $(\eta_0, u_0) \in H^s(\mathbb{R})^2$*, there is a* $T > 0$ *and a unique solution* (η, u) *of (3.31) that satisfies* $(\eta, u) \in C(0, T; H^s(\mathbb{R})^2)$*. Additionally,* $(\eta_t, u_t) \in C(0, T; H^{s-1}(\mathbb{R})^2)$*. Moreover, the correspondence* $(\eta_0, u_0) \mapsto (\eta, u)$ *is locally Lipschitz continuous.*

- Case II: $\mathcal{H}$ has order -1, i.e., $a = 0, b > 0, c < 0, d > 0$.

  Theorem 7. *Let $s \geq 0$ and $(\eta_0, u_0) \in H^{s+1} \times H^s$, there is a $T > 0$ and a unique solution (η, u) of (3.31) that satisfies $(\eta, u) \in C(0, T; H^{s+1}(\mathbb{R})) \times C(0, T; H^s(\mathbb{R}))$. Additionally, $(\eta_t, u_t) \in C(0, T; H^{s+1}(\mathbb{R})) \times C(0, T; H^s(\mathbb{R}))$. Moreover, the correspondence $(\eta_0, u_0) \mapsto (\eta, u)$ is locally Lipschitz.*

- Case III: $\mathcal{H}$ has order 1, i.e., $a < 0, b > 0, c = 0, d > 0$.

  Theorem 8. *Let $s \geq 0$ and $(\eta_0, u_0) \in H^s \times H^{s+1}$, there is a $T > 0$ and a unique solution (η, u) of (3.31) that satisfies $(\eta, u) \in C(0, T; H^s(\mathbb{R})) \times C(0, T; H^{s+1}(\mathbb{R}))$. Additionally, $(\eta_t, u_t) \in C(0, T; H^s(\mathbb{R})) \times C(0, T; H^{s+1}(\mathbb{R}))$. Moreover, the correspondence $(\eta_0, u_0) \mapsto (\eta, u)$ is locally Lipschitz.*

(iii) Purely KdV-type Boussinesq systems: **$b = d = 0, a \neq 0, c \neq 0$.**

The only admissible case is when $a = c > 0$, which means $\theta^2 = \frac{2}{3}$, and $a = c = \frac{1}{6}$.

Theorem 9. *Let $s > \frac{3}{4}$. For any $(\eta_0, u_0) \in H^s(\mathbb{R})^2$, there is a $T > 0$ and a unique solution (η, u) of (3.31) that satisfies $(\eta, u) \in C(0, T; H^s(\mathbb{R})^2)$. Additionally, $(\eta_t, u_t) \in C(0, T; H^{s-3}(\mathbb{R})^2)$. Moreover, the correspondence $(\eta_0, u_0) \mapsto (\eta, u)$ is analytic.*

(iv) The Boussinesq system when $\mathcal{H}$ has order 2: **$a < 0, b = 0, c = 0, d > 0$.**

Theorem 10. *Let $s \geq 1$. For any $(\eta_0, u_0) \in H^s(\mathbb{R}) \times H^{s+2}(\mathbb{R})$, there is a $T > 0$ and a unique solution (η, u) of (3.31) that satisfies $(\eta, u) \in C(0, T; H^s(\mathbb{R}) \times H^{s+2}(\mathbb{R}))$. Additionally, $(\eta_t, u_t) \in C(0, T; H^{s-1}(\mathbb{R}) \times H^{s+1}(\mathbb{R}))$. The solution depends continuously upon perturbations of the initial data in the relevant function classes.*

(v) The Boussinesq system when $\mathcal{H}$ has order 1: $a = c \geq 0, b = 0, d > 0$ ***and*** $a < 0, b = 0, c < 0, d > 0$.

Theorem 11. *Let $s \geq 1$. For any $(\eta_0, u_0) \in H^s(\mathbb{R}) \times H^{s+1}(\mathbb{R})$, there is a $T > 0$ and a unique solution (η, u) of (3.31) that satisfies $(\eta, u) \in C(0, T; H^s(\mathbb{R}) \times H^{s+1}(\mathbb{R}))$. Additionally, $(\eta_t, u_t) \in C(0, T; H^{s-2}(\mathbb{R}) \times H^{s-1}(\mathbb{R}))$. The mapping of initial data to its associated solution is continuous.*

Theorem 12. *Let $s \geq 1$. For any $(\eta_0, u_0) \in H^s(\mathbb{R}) \times H^{s+1}(\mathbb{R})$ such that $inf_{x\in\mathbb{R}} \{1 + \eta_0(x)\} > 0$, there is a $T > 0$ and a unique solution (η, u) of (3.31) that satisfies $(\eta, u) \in C(0, T; H^s(\mathbb{R}) \times H^{s+1}(\mathbb{R}))$. Additionally, $(\eta_t, u_t) \in C(0, T; H^{s-2}(\mathbb{R}) \times H^{s-1}(\mathbb{R}))$. Moreover, given $r > 0$ the solution depends continuously upon the initial data in the class*

$$\mathcal{H}^s_r = \left\{(\eta_0, u_0) \in H^s(\mathbb{R}) \times H^{s+1}(\mathbb{R}) : 1 + \eta_0(x) > r \ \forall \ x\right\}$$

Remark 3. *The second theorem of this topic is related with the Classical Boussinesq system. The parameters in this case are $a = c = b = 0$ and $d = \frac{1}{3}$.*

(vi) The Boussinesq system when $\mathcal{H}$ has order 0: $a < 0, b > 0, c = d = 0$ ***and*** $a = b = 0, c < 0, d > 0$.

Theorem 13. *Let $s \geq 2$. For any $(\eta_0, u_0) \in H^s(\mathbb{R})^2$, there is a $T > 0$ and a unique solution (η, u) of (3.31) that satisfies $(\eta, u) \in C(0, T; H^s(\mathbb{R})^2)$. Additionally, $(\eta_t, u_t) \in C(0, T; H^{s-1}(\mathbb{R})^2)$. The mapping of initial data to its associated solution is continuous from the space $H^s(\mathbb{R}) \times H^s(\mathbb{R})$ to $C(0, T; H^s(\mathbb{R})^2)$.*

(vii) The Boussinesq system when $\mathcal{H}$ has order -1: $a = c \geq 0, b > 0, d = 0$.

Theorem 14. *Let $s \geq 1$. For any $(\eta_0, u_0) \in H^{s+1}(\mathbb{R}) \times H^s(\mathbb{R})$, there is a $T > 0$ and a unique solution (η, u) of (3.31) that satisfies $(\eta, u) \in C(0, T; H^{s+1}(\mathbb{R}) \times H^s(\mathbb{R}))$.*

Additionally, $(\eta_t, u_t) \in C(0,T;H^{s-1}(\mathbb{R}) \times H^{s-2}(\mathbb{R}))$*. The mapping of initial data to its associated solution is continuous.*

Remark 4. *It is worth noticing that all the results presented in this section are equally valid in the case of periodic boundary conditions by considering the corresponding periodic Sobolev spaces.*

4 NUMERICAL ANALYSIS

In this chapter, we perform the stability and convergence analysis of the numerical methods used to find the numerical solution of the Boussinesq systems families given by (3.27) and (3.31), the linear and nonlinear case, respectively. The discrete equations are obtained by applying a Fourier collocation method in space and the explicit RK4 method in time. This analysis shall be used as a guidance on the choice of the time step size for the numerical simulations. To simplify the notations, we consider the case when the solution is 2π-periodic in space.

4.1 Fully discretized Boussinesq system

The Fourier collocation method can be formulated using the discrete Fourier transform (DFT) as we introduced in chapter 2. We start by presenting the notation and relevant definitions.

Let $N > 0$ be a fixed even integer and define the grid points $x_j = 2\pi j/N$, $j = 0, \ldots, N-1$. Given a 2π-periodic real function g, the vector of sampled values at the grid points is given by

$$\mathbf{g} = [g_0, \ldots, g_{N-1}]^t \in \mathbb{R}^N,$$

with $g_j = g(x_j)$ and its discrete Fourier transform is defined as

$$\hat{\mathbf{g}} = [\hat{g}_{-\frac{N}{2}}, \ldots, \hat{g}_{\frac{N}{2}-1}]^t \in \mathbb{C}^N,$$

where

$$\hat{g}_k = \frac{1}{N}\sum_{j=0}^{N-1} g_j e^{-ikx_j}, \quad \text{for } k = -N/2, \ldots, N/2-1. \tag{4.1}$$

One can recover the sampled vector using the inverse discrete Fourier transform (IDFT)

$$g_j = \sum_{k=-N/2}^{N/2-1} \hat{g}_k e^{ikx_j}, \quad \text{for } j = 0, \ldots, N-1. \tag{4.2}$$

Based on this equation, we can define a trigonometric polynomial $g_N(x) = (P_N \mathbf{g})(x)$ that interpolates $g(x)$ at the grid points x_j, $j = 0, \ldots, N-1$.

The Fourier collocation method is obtained by requiring the differential equations to be satisfied at the grid points when the involved functions are substituted by their interpolators (see CANUTO et al. (2006) and TREFETHEN (2000) for details). The discretized equations corresponding to the nonlinear Boussinesq system (3.31) are given by

$$\begin{aligned} \eta_{N_t} + u_{N_x} + ((u\eta)_N)_x + au_{N_{xxx}} - b\eta_{N_{xxt}}\Big|_{x=x_j} &= 0, \\ u_{N_t} + \eta_{N_x} + \frac{1}{2}\left((u^2)_N\right)_x + c\eta_{N_{xxx}} - du_{N_{xxt}}\Big|_{x=x_j} &= 0. \end{aligned} \tag{4.3}$$

Applying the DFT, we get the following system of ordinary differential equations (ODEs)

$$\begin{aligned} \hat{\boldsymbol{\eta}}_t &= \mathbf{f}(\hat{\boldsymbol{\eta}}, \hat{\mathbf{u}}), \\ \hat{\mathbf{u}}_t &= \mathbf{g}(\hat{\boldsymbol{\eta}}, \hat{\mathbf{u}}), \end{aligned} \tag{4.4}$$

where $\boldsymbol{\eta}(t) \approx [\eta(t, x_0), \ldots, \eta(t, x_{N-1})]^t$, $\mathbf{u}(t) \approx [u(t, x_0), \ldots, u(t, x_{N-1})]^t$,

$$(\mathbf{f}(\hat{\boldsymbol{\eta}}, \hat{\mathbf{u}}))_k = \begin{cases} 0, & \text{if } k = -\frac{N}{2}, \\ -ikw_1(k)\hat{u}_k - \frac{ik}{1+bk^2}\widehat{(\boldsymbol{\eta} \circ \mathbf{u})}_k, & \text{otherwise}, \end{cases}$$

$$(\mathbf{g}(\hat{\boldsymbol{\eta}}, \hat{\mathbf{u}}))_k = \begin{cases} 0, & \text{if } k = -\frac{N}{2}, \\ -ikw_2(k)\hat{\eta}_k - \frac{ik}{2(1+dk^2)}\widehat{(\mathbf{u} \circ \mathbf{u})}_k, & \text{otherwise}. \end{cases}$$

We use the symbol "$\circ$"to represent the Hadamard product of two vectors (entry-wise multiplication). The especial treatment of the Fourier mode corresponding to $k = -N/2$ is the result of the asymmetry in equation (4.2) regarding this wavenumber, as we discussed in section 2.2.2.

In order to develop the fully discrete scheme to obtain approximate solutions of the system of $2N$ ODEs (4.4) in a time interval $[0,T]$, we use the RK4 method introduced in chapter 2. From the corresponding initial conditions we obtain the approximations $\hat{\boldsymbol{\eta}}^n \approx \hat{\boldsymbol{\eta}}(t_n)$, $\hat{\mathbf{u}}^n \approx \hat{\mathbf{u}}(t_n)$ where t_n, $n=0,\ldots,M$ represents a discretization of the time interval $[0,T]$. Furthermore, applying the IDFT we arrive at the approximations

$$\boldsymbol{\eta}_j^n \approx \eta(t_n,x_j),\ \mathbf{u}_j^n \approx u(t_n,x_j), \quad n=1\ldots,M,\ j=0,\ldots,N-1.$$

It is worth noticing that P_N denote the L^2-orthogonal projection onto S_N. This projection has the following approximation properties, whose proofs is standard (see CANUTO et al. (1987) and MERCIER (1983)).

Proposition 4. *Given integers $0 \leq s \leq r$, there exists a constant C independent of N such that, for any $f \in H^r$,*

$$\|f - P_N f\|_s \leq C N^{s-r} \|f\|_r. \tag{4.5}$$

Moreover, as a consequence of Sobolev lemma given in chapter 2 and the last inequality, it holds that for any $f \in H^r$, $r \geq 1$

$$\|f - P_N f\|_\infty \leq C N^{\frac{1}{2}-r} \|f\|_r. \tag{4.6}$$

There also are the inverse inequalities on S_N: given $0 \leq s \leq r$, there is a constant C_0 independent of N, such that for all ψ in S_N,

$$\|\psi\|_r \leq C_0 N^{r-s} \|\psi\|_s, \qquad \|\psi\|_\infty \leq C_0 N^{1/2} \|\psi\|. \tag{4.7}$$

4.2 Stability Analysis of the Linear Family Systems

In this section, we carry out the von Neumann analysis of the fully discretized scheme associated with the linear system (3.27), which gives rise to a similar system to (4.4), in the absence of the terms containing Hadamard products.

The semi discretized equations corresponding to Boussinesq system (3.27) are similar to equations in (4.3), in the absence of the terms $((u\eta)_N)_x$ and $\frac{1}{2}((u^2)_N)_x$. Then, following ALFARO VIGO et al. (2014), we can represent the time discretization by RK4 method in Taylor expansion form as

$$\begin{aligned} \hat{\eta}_k^{n+1} = & \left\{1 - k^2 w_1(k) w_2(k) \frac{\Delta t^2}{2!} + k^4 w_1^2(k) w_2^2(k) \frac{\Delta t^4}{4!}\right\} \hat{\eta}_k^n \\ & + \left\{-ikw_1(k)\Delta t + ik^3 w_1^2(k) w_2(k) \frac{\Delta t^3}{3!}\right\} \hat{u}_k^n, \end{aligned} \tag{4.8}$$

$$\begin{aligned} \hat{u}_k^{n+1} = & \left\{1 - k^2 w_1(k) w_2(k) \frac{\Delta t^2}{2!} + k^4 w_1^2(k) w_2^2(k) \frac{\Delta t^4}{4!}\right\} \hat{u}_k^n \\ & + \left\{-ikw_2(k)\Delta t + ik^3 w_1(k) w_2^2(k) \frac{\Delta t^3}{3!}\right\} \hat{\eta}_k^n, \end{aligned} \tag{4.9}$$

where the superscripts indicate the functions $\hat{\eta}_k(t)$ and $\hat{u}_k(t)$ evaluated at the respective point of the time mesh.

The equations (4.8)-(4.9) can be represented in matrix form by $\left[\hat{\eta}_k^{n+1} \ \hat{u}_k^{n+1}\right]^T = G_k \left[\hat{\eta}_k^n \ \hat{u}_k^n\right]^T$, where

$$G_k = \begin{bmatrix} A(k) & B(k) \\ C(k) & A(k) \end{bmatrix}, \tag{4.10}$$

is the amplification matrix corresponding to the mode with Fourier number k and

$$\begin{aligned} A(k) &= 1 - k^2 w_1(k) w_2(k) \tfrac{\Delta t^2}{2!} + k^4 w_1^2(k) w_2^2(k) \tfrac{\Delta t^4}{4!}, \\ B(k) &= -ikw_1(k)\Delta t + ik^3 w_1^2(k) w_2(k) \tfrac{\Delta t^3}{3!}, \\ C(k) &= -ikw_2(k)\Delta t + ik^3 w_1(k) w_2^2(k) \tfrac{\Delta t^3}{3!}. \end{aligned}$$

We start the von Neumann analysis searching for solutions of the form $[\hat{\eta}_k^n \; \hat{u}_k^n]^T = g_k^n \, [a_k \; b_k]^T$, with $[a_k \; b_k]^T \neq 0$. Therefore, we obtain

$$g_k^{n+1} \begin{bmatrix} a_k \\ b_k \end{bmatrix} = G_k \; g_k^n \begin{bmatrix} a_k \\ b_k \end{bmatrix}, \tag{4.11}$$

which implies that, dividing both sides of the (4.11) by g_k^n, $\det(G_k - I g_k) = 0$, since $[a_k \; b_k]^T \neq 0$ by hypothesis.

Using the definition of G_k given by (4.10) and the fact that $G_k - I g_k$ is a singular matrix, we obtain that

$$(A(k) - g_k)^2 = \alpha(k), \tag{4.12}$$

for each $k = \{-N/2 + 1, \ldots, N/2\}$, where

$$\alpha(k) = -k^2 w_1(k) w_2(k) \Delta t^2 + \frac{k^4}{3} w_1^2(k) w_2^2(k) \Delta t^4 - \frac{k^6}{36} w_1^3(k) w_2^3(k) \Delta t^6.$$

The equation (4.12) implies that the amplification factors are given by

$$g_k^{\pm} = A(k) \pm \sqrt{\alpha(k)}. \tag{4.13}$$

The stability of the solution for the fully discrete equation (4.9), according to von Neumann analysis, is sufficiently guaranteed if $|g_k| \leq 1$ for all k. Then, using (4.13), it has to be satisfied that

$$|A(k) \pm \sqrt{\alpha(k)}| \leq 1, \quad \text{for all} \quad k.$$

Observe that,

$$\begin{aligned} \alpha(k) =& - k^2 w_1(k) w_2(k) \Delta t^2 + \frac{k^4}{3} w_1^2(k) w_2^2(k) \Delta t^4 - \frac{k^6}{36} w_1^3(k) w_2^3(k) \Delta t^6 \\ =& - \frac{1}{36} k^2 \Delta t^2 w_1(k) w_2(k) \Big(k^2 w_1(k) w_2(k) \Delta t^2 - 6 \Big)^2, \end{aligned}$$

which implies that

$$\sqrt{\alpha(k)} = i\frac{|k|\Delta t}{6}\left|k^2 w_1(k) w_2(k)\Delta t^2 - 6\right|\sqrt{w_1(k)w_2(k)}.$$

Then,

$$\begin{aligned}|g_k^{\pm}|^2 &= \left|A(k) \pm i\frac{|k|\Delta t}{6}|k^2 w_1(k)w_2(k)\Delta t^2 - 6|\sqrt{w_1(k)w_2(k)}\right|^2 \\ &= A^2(k) + \frac{k^2\Delta t^2}{36}\left(k^2 w_1(k)w_2(k)\Delta t^2 - 6\right)^2 w_1(k)w_2(k) \\ &= 1 - \frac{k^6}{72}w_1^3(k)w_2^3(k)\Delta t^6 + \frac{k^8}{576}w_1^4(k)w_2^4(k)\Delta t^8.\end{aligned}$$

If we take $y = k^2 w_1(k)w_2(k)\Delta t^2$, we get

$$|g_k^{\pm}|^2 - 1 = p(y),$$

where $p(y) = \frac{y^3}{576}(y-8)$. To ensure stability, as it was mentioned, we seek for a relation between Δt and N such that $|g_k^{\pm}| \leq 1$, or equivalently, determine y such that $p(y) \leq 0$. Then,

$$\frac{y^3}{576}(y-8) \leq 0 \quad \Leftrightarrow \quad 0 \leq y \leq 8.$$

Note that if $k = 0$ there is no restriction for Δt. Therefore, for $k = \{-N/2 + 1, \ldots, N/2\}$, with $k \neq 0$, it has to be satisfied that

$$0 \leq k^2 w_1(k)w_2(k)\Delta t^2 \leq 8 \quad \Leftrightarrow \quad 0 \leq \Delta t \leq \sqrt{8}\left(|\omega(k)|\right)^{-1}, \tag{4.14}$$

with $\omega(k) = k\sigma(k) = k\sqrt{w_1(k)w_2(k)}$.

To relate Δt with N, we have to analyze the function $|\omega(k)|$. It is worth to observe that since $|\omega(k)|$ is an even function, we will only analyze this function for $k \geq 1$.

Thus, for $k \geq 1$, we are considering $\omega(k)$ as a function which depends on the parameters a, b, c, d given by (1.13), such that

$$\omega(k) = k\sqrt{\frac{(1-ak^2)(1-ck^2)}{(1+bk^2)(1+dk^2)}}, \tag{4.15}$$

and we will be interested in analyzing the behavior of this function as k approaches infinity.

As mentioned in the section 3.3, the linear Boussinesq system (3.27) is well posed if its parameters lay in the in three different regions that were defined in Proposition 3. We have the following possibilities:

C1: In this case $b \geq 0,\ d \geq 0,\ a \leq 0,\ c \leq 0$, which implies the following asymptotic behaviors for $\omega(k)$

<table>
<tr><th></th><th></th><th>$\omega(k)$ at ∞</th><th></th><th></th><th>$\omega(k)$ at ∞</th></tr>
<tr><td rowspan="4">$a<0$
$b>0$</td><td>$c<0, d>0$</td><td>c_1k^1</td><td rowspan="4">$a<0$
$b=0$</td><td>$c<0, d>0$</td><td>c_9k^2</td></tr>
<tr><td>$c=0, d>0$</td><td>c_2k^0</td><td>$c=0, d>0$</td><td>$c_{10}k^1$</td></tr>
<tr><td>$c<0, d=0$</td><td>c_3k^2</td><td>$c<0, d=0$</td><td>$c_{11}k^3$</td></tr>
<tr><td>$c=0, d=0$</td><td>c_4k^1</td><td>$c=0, d=0$</td><td>$c_{12}k^2$</td></tr>
<tr><td rowspan="4">$a=0$
$b>0$</td><td>$c<0, d>0$</td><td>c_5k^0</td><td rowspan="4">$a=0$
$b=0$</td><td>$c<0, d>0$</td><td>$c_{13}k^1$</td></tr>
<tr><td>$c=0, d>0$</td><td>c_6k^{-1}</td><td>$c=0, d>0$</td><td>$c_{14}k^0$</td></tr>
<tr><td>$c<0, d=0$</td><td>c_7k^1</td><td>$c<0, d=0$</td><td>$c_{15}k^2$</td></tr>
<tr><td>$c=0, d=0$</td><td>c_8k^0</td><td>$c=0, d=0$</td><td>$c_{16}k^1$</td></tr>
</table>

C2: In this case $b \geq 0,\ d \geq 0,\ a = c > 0$, which implies the following asymptotic behaviors for $\omega(k)$

<table>
<tr><th></th><th></th><th>$\omega(k)$ at ∞</th></tr>
<tr><td rowspan="4">$a=c>0$</td><td>$b>0, d>0$</td><td>$c_{17}k^1$</td></tr>
<tr><td>$b=0, d>0$</td><td>$c_{18}k^2$</td></tr>
<tr><td>$b>0, d=0$</td><td>$c_{19}k^2$</td></tr>
<tr><td>$b=0, d=0$</td><td>$c_{20}k^3$</td></tr>
</table>

C3: In this case $b = d < 0$, $a = c > 0$, which implies the following asymptotic behavior for $\omega(k)$

		$\omega(k)$ at ∞
$a = c > 0$	$b = d < 0$	$c_{21}k^1$

Then, we conclude that for each of the 21 cases described above, there exist a positive constant c_i, $i = 1, \dots, 21$, and $p_i \in \{-1, 0, 1, 2, 3\}$ such that

$$\omega(k) \;\sim\; c_i k^{p_i} \quad \text{as} \quad k \to \infty. \tag{4.16}$$

Using (4.16), we can guarantee that for each of these cases, given $\varepsilon > 0$ there exists $K_0 = K_0(\varepsilon, a, b, c, d)$ such that

$$|\omega(k)| \le c_i(1+\varepsilon)k^{p_i} \quad \text{for all} \quad k \ge K_0. \tag{4.17}$$

On the other hand, if we consider $\tilde{c}_i = \max\left\{\frac{|\omega(k)|}{k^{p_i}} : 1 \le k \le K_0\right\}$, taking $C = \max\ \{\tilde{c}_i, (1+\varepsilon)c_i\}$ we have by (4.17) that

$$|\omega(k)| \le Ck^{p_i} \quad \text{for all} \quad k \ge 1, \tag{4.18}$$

where $p_i = \{-1, 0, 1, 2, 3\}$ varies according to the choices of a, b, c, d.

However, since the cases when $p_i = -1$ turns the right side of (4.18) decreasing with k, we set $\ell = \max\{0, p_i\}$ and therefore, it holds that

$$|\omega(k)| \le Ck^{\ell} \quad \text{for all} \quad k \ge 1, \tag{4.19}$$

with $\ell \in \{0, 1, 2, 3\}$ depending on the choice of a, b, c, d.

Using (4.19), we finally obtain that

$$\max_{k\in[1,\frac{N}{2}]}\Big(|\omega(k)|\Big) \le \max_{k\in[1,\frac{N}{2}]} Ck^{\ell} \le \tilde{C}N^{\ell}. \tag{4.20}$$

Hence, the equation (4.20) implies that

$$\frac{\sqrt{8}}{|\omega(k)|} \ge \frac{\sqrt{8}}{\max\limits_{k\in[1,\frac{N}{2}]}\Big(|\omega(k)|\Big)} \ge \overline{C}N^{-\ell} \tag{4.21}$$

Since we want to satisfy the equation (4.14), from (4.21) we conclude that it is sufficient to satisfy the stability condition that $\Delta t \le \overline{C}N^{-\ell}$ for some positive constant $\overline{C}$ and $\ell \in \{0,1,2,3\}$ depending on the parameters a,b,c,d that have been considered. The corresponding values of ℓ can be observed in Table 4.1.

$\ell = 0$	$\ell = 1$
$a<0, b>0, c=0, d>0$	$a<0, b>0, c<0, d>0$
$a=0, b>0, c<0, d>0$	$a<0, b>0, c=0, d=0$
$a=0, b>0, c=0, d=0$	$a=0, b>0, c<0, d=0$
$a=0, b>0, c=0, d>0$	$a<0, b=0, c=0, d>0$
$a=0, b=0, c=0, d>0$	$a=0, b=0, c<0, d>0$
	$a=0, b=0, c=0, d=0$
	$a=c>0, b>0, d>0$
	$a=c>0, b=d<0$
$\ell = 2$	$\ell = 3$
$a<0, b>0, c<0, d=0$	$a<0, b=0, c<0, d=0$
$a<0, b=0, c<0, d>0$	$a=c>0, b=0, d=0$
$a<0, b=0, c=0, d=0$	
$a=0, b=0, c<0, d=0$	
$a=c>0, b=0, d>0$	
$a=c>0, b>0, d=0$	

Table 4.1: Values of ℓ corresponding to the parameters a,b,c,d.

Observe that the cases where $\ell = 0$ or $\ell = 1$ are the most interesting cases since the stability requirements are not too restrictive. By *not too restrictive* we mean that the dependence between the spacial and time steps sizes does not generate stiff problems. Therefore, the corresponding fully-discrete schemes can achieve a high accuracy at a low cost in regard to the computational time.

4.3 Convergence Analysis of the Nonlinear Family Systems

In this section, we perform the convergence analysis for the nonlinear family of Boussinesq systems (3.31). We start considering a_B and a_D two elliptic projection onto S_N defined through P_N by

$$a_B(P_N v, \xi) = a_B(v, \xi), \qquad a_D(P_N w, \xi) = a_D(w, \xi), \tag{4.22}$$

for all $\xi \in S_N$, where $a_B, a_D : H^1 \times H^1 \to \mathbb{R}$ are two bilinear forms defined by

$$\begin{aligned} a_B(u, v) &:= (u, v) + b(u_x, v_x), \\ a_D(u, v) &:= (u, v) + d(u_x, v_x). \end{aligned} \tag{4.23}$$

Proposition 5. *For $b, d \geq 0$, a_B and a_D are coercives in their domains of definition.*

Proof. For $b > 0$, using the definition of a_B we have

$$a_B(v, v) = \int_\Omega v^2 \; dS + \int_\Omega v_x^2 \; dS \geq \min\{1, b\} \int_\Omega (v^2 + v_x^2) \; dS,$$

which implies that $a_B(v, v) \geq C\|v\|_1^2$. The proof is analogous for a_D with $d > 0$. □

The convergence analysis is divided in cases which the results about well posedness for the nonlinear problem are known, as we established in section 3.3.2.

4.3.1 Semidiscrete Problem

In this section, considering the time variable continuous, we prove optimal order L^2, H^1, H^2 error bounds of spectral accuracy in $\Omega = [-L, L]$ for the nonlinear family of Boussinesq systems (3.31).

4.3.1.1 Purely BBM-type Boussinesq System

For this case, we assume that $b, d > 0$ and $a = c = 0$. Then, the Boussinesq system becomes:

$$\begin{aligned} \eta_t + u_x + (\eta u)_x - b\eta_{xxt} = 0, \\ u_t + \eta_x + uu_x - du_{xxt} = 0, \end{aligned} \tag{4.24}$$

for $x \in \Omega$ and $t \geq 0$, supplemented with initial conditions $\eta(x, 0) = \eta_0(x)$ and $u(x, 0) = u_0(x)$.

In order to obtain error bounds for the semi discrete problem obtained from (4.24) using the Fourier collocation method, we consider weight functions, namely φ, ψ, as functions from S_N as well, and require that the following system is satisfied

$$\begin{aligned} (\eta_{N_t}, \varphi) + (u_{N_x}, \varphi) + ((\eta_N u_N)_x, \varphi) - b(\eta_{N_{xxt}}, \varphi) = 0, \\ (u_{N_t}, \psi) + (\eta_{N_x}, \psi) + (u_N u_{N_x}, \psi) - d(u_{N_{xxt}}, \psi) = 0, \end{aligned} \tag{4.25}$$

in Ω and for $t \geq 0$. The system (4.25) is supplemented with the initial conditions $\eta_N(x, 0) = P_N\eta_0(x)$ and $u_N(x, 0) = P_N u_0(x)$, for $x \in \Omega$.

Moreover, as it was seen on the Section 3.3.2, the inital condition and the solution in this case satisfy that $(\eta_0, u_0) \in H^s(\Omega) \times H^s(\Omega)$ and $(\eta, u) \in C(0, T; H^s(\Omega) \times H^s(\Omega))$, for $s \geq 0$ and $T > 0$.

We initiate this analysis giving an estimate between the exact solution (η, u) of (4.24) and (V, S), solution of a linearized version of (4.24) that it is defined in the following. After that, we are able to estimate the error between (η, u) and (η_N, u_N), solution of (4.25).

For the first part, we linearize (4.25) as follows: given $(\eta, u) \in C(0, T; H^s(\Omega)\times H^s(\Omega))$ solution of (4.24) corresponding to initial data $(\eta_0, u_0) \in H^s(\Omega) \times H^s(\Omega)$, we seek for functions $V, S \in S_N$, which for all $\varphi, \psi \in S_N$ and for all $t \geq 0$, satisfy

$$\begin{aligned} (V_t, \varphi) + (S_x, \varphi) + ((uV_x + \eta S_x), \varphi) - b(V_{xxt}, \varphi) = 0, \\ (S_t, \psi) + (V_x, \psi) + (uS_x, \psi) - d(S_{xxt}, \psi) = 0, \end{aligned} \tag{4.26}$$

with initial conditions given by $V(x, 0) = P_N\eta_0(x)$ and $S(x, 0) = P_N u_0(x)$ for $x \in \Omega$.

We have the following result:

Lemma 3. *Let $(\eta, u) \in C(0, T; H^s(\Omega) \times H^s(\Omega))$ be the solution of (4.24) corresponding to initial condition $(\eta_0, u_0) \in H^s(\Omega) \times H^s(\Omega)$, $s \geq 2$. Then, there is a unique solution $(V, S) \in S_N$ of (4.26) for all finite time $T \geq 0$. Moreover, given $0 \leq t \leq T$, there is a constant $C = C(T, b, d) > 0$ independent of N, such that*

$$\max_{0\leq t\leq T} \{\|\eta(t) - V(t)\|_1 + \|u(t) - S(t)\|_1\} \leq CN^{1-s}. \tag{4.27}$$

Proof. Let $\rho = P_N\eta - V$ and $\theta = P_N u - S$. Then, it holds that

$$\begin{aligned} \eta - V = \eta - P_N\eta + P_N\eta - V = \eta - P_N\eta + \rho, \\ u - S = u - P_N u + P_N u - S = u - P_N u + \theta. \end{aligned} \tag{4.28}$$

To obtain (4.27) using (4.28), we only have to estimate the terms ρ and θ, since the estimates for the terms $\eta - P_N\eta$ and $u - P_N u$ can be obtained by (4.5).

Then, to obtain these estimates, we consider the following formulation for this case

$$\begin{aligned} (\eta_t, \varphi) + (u_x, \varphi) + ((\eta u)_x, \varphi) - b(\eta_{xxt}, \varphi) = 0, \\ (u_t, \psi) + (\eta_x, \psi) + (uu_x, \psi) - d(u_{xxt}, \psi) = 0, \end{aligned}$$

such that, subtracting it from (4.26), implies

$$\begin{aligned} ((\eta - V)_t, \varphi) + ((u - S)_x, \varphi) + ((\eta u)_x - uV_x - \eta S_x, \varphi) - b((\eta - V)_{xxt}, \varphi) = 0, \\ ((u - S)_t, \psi) + ((\eta - V)_x, \psi) + (uu_x - uS_x, \psi) - d((u - S)_{xxt}, \psi) = 0. \end{aligned} \tag{4.29}$$

If we note that,

$$\begin{aligned} (\eta u)_x - uV_x - \eta S_x &= \eta_x u + \eta u_x - uV_x - \eta S_x \\ &= u(\eta_x - V_x) + \eta(u_x - S_x) \\ &= u(\eta - P_N\eta)_x + u\rho_x + \eta(u - P_N u)_x + \eta\theta_x, \end{aligned} \tag{4.30}$$

and

$$\begin{aligned} uu_x - uS_x &= u(u_x - S_x) \\ &= u(u - P_N u)_x + u\theta_x, \end{aligned} \tag{4.31}$$

we can use the property of the L^2-orthogonal projection P_N and the definitions of a_B, a_D in (4.23), to rewrite (4.29) as

$$\begin{aligned} a_B(\rho_t, \varphi) + (\theta_x, \varphi) + (u(\eta - P_N\eta)_x + u\rho_x + \eta(u - P_N u)_x + \eta\theta_x, \varphi) = 0, \\ a_D(\theta_t, \psi) + (\rho_x, \psi) + (u(u - P_N u)_x + u\theta_x, \psi) = 0, \end{aligned} \tag{4.32}$$

supplemented with initial conditions $\rho(0) = \theta(0) = 0$.

Taking $\varphi = \rho$ and $\psi = \theta$ in (4.32), we have from the first equation that

$$a_B(\rho_t, \rho) = -(\theta_x, \rho) - (u(\eta - P_N\eta)_x, \rho) - (u\rho_x, \rho) - (\eta(u - P_N u)_x, \rho) - (\eta\theta_x, \rho), \tag{4.33}$$

and from the second equation that,

$$a_D(\theta_t, \theta) = -(\rho_x, \theta) - (u(u - P_N u)_x, \theta) - (u\theta_x, \theta), \tag{4.34}$$

Observing that $a_B(\rho_t, \rho) = \frac{1}{2}\frac{d}{dt} a_B(\rho, \rho)$ (also valid for a_D evaluated at (θ_t, θ)), we can add (4.33) and (4.34) and use the coercivity of a_B and a_D along with the Cauchy Schwarz and Young inequalities presented in chapter 2, to obtain that

$$\frac{d}{dt}\left(\|\rho\|_1^2 + \|\theta\|_1^2\right) \leq k_1 + k_2\left(\|\rho\|_1^2 + \|\theta\|_1^2\right), \tag{4.35}$$

where $k_1 = \|u\|_\infty^2 \|(\eta - P_N\eta)_x\|^2 + \|\eta\|_\infty^2 \|(u - P_N u)_x\|^2 + \|u\|_\infty^2 \|(u - P_N u)_x\|^2$ and $k_2 = \max\{4 + \|u_x\|_\infty, 1, 2 + \|u_x\|_\infty, 1 + \|\eta\|_\infty^2\}$.

On the other hand, since $(\eta, u) \in C(0,T; H^s(\Omega) \times H^s(\Omega))$ for $s \geq 0$, from the Sobolev lemma, it holds that

$$\begin{aligned}
&\|\eta(t)\|_\infty \leq C_s\|\eta\|_1 \leq C\|\eta\|_s \leq C, \\
&\|u(t)\|_\infty \leq C_s\|u\|_1 \leq C\|u\|_s \leq C, \\
&\|\eta_x(t)\|_\infty \leq C_s\|\eta_x\|_1 \leq C\|\eta\|_2 \leq C\|\eta\|_s \leq C, \\
&\|u_x(t)\|_\infty \leq C_s\|u_x\|_1 \leq C\|u\|_2 \leq C\|u\|_s \leq C,
\end{aligned} \tag{4.36}$$

if we consider $s \geq 2$. The constant C_s represents the Sobolev constant, and the other embedding constants are all represented by C to simplify the notation.

The estimates given in (4.36) and (4.5), lead us to conclude that $k_1 \leq CN^{2(1-s)}$ and that k_2 is bounded. Then, integrating (4.35) over $[0,t]$ and applying the Gronwall's lemma, we get

$$\|\rho(t)\|_1 + \|\theta(t)\|_1 \leq C(T,b,d)N^{1-s}. \tag{4.37}$$

The estimate (4.37) along with the decomposition given by (4.28) finish the proof. $\square$

Remark 5. *Observe that to obtain (4.27), we supposed that (4.26) had a unique solution for all $t \in [0,T]$. The existence and uniqueness of a local solution can be proved through the theory of ODE systems. Indeed, if we take $\varphi = \psi = e^{ikx}$ for $k = -N, \ldots, N$ in (4.26), and observe that, since $(V,S) \in S_N$, each component can be represented by a combination of its Fourier coefficients $\hat{V}$ and $\hat{S}$, we obtain a linear ODE system given by*

$$\begin{aligned}
\hat{V}_t(k,t) &= \tfrac{-ik}{1+bk^2}\left(\hat{S} + u\hat{V} + \eta\hat{S}\right), \\
\hat{S}_t(k,t) &= \tfrac{-ik}{1+dk^2}\left(\hat{V} + u\hat{S}\right),
\end{aligned} \tag{4.38}$$

with $\hat{V}(k,0)=\hat{\eta_0}(k)$ *and* $\hat{S}(k,0)=\hat{u_0}(k)$.

The right hand side of (4.38) is locally Lipschitz continuous, with respect to ℓ^2 *norm (Euclidean norm). Therefore, the existence of a maximal time* t_h, $0<t_h<T$ *such that, for all* $t<t_h$ *there exists a unique pair of solutions* (V,S) *of (4.26) is a classical result of the theory of ODEs systems.*

Thus, to finish this proof, we just need to show that this solution does not blow up if we extend it over $[0,T]$. *With this aim in mind, we resort to a stability result in* L^2*-norm. Taking* $\varphi=V$ *and* $\psi=S$ *in (4.26) we have from the first equation that*

$$a_B(V_t,V)=-2(S_x,V)+(u_x,V^2)-2(\eta S_x,V), \tag{4.39}$$

and, from the second equation, that

$$a_D(S_t,S)=-2(V_x,S)+(u_x,S^2). \tag{4.40}$$

Adding (4.39) and (4.40) and proceeding as before, we get that

$$\tfrac{d}{dt}\left(\|V\|_1^2+\|S\|_1^2\right)\leq K\left(\|V\|_1^2+\|S\|_1^2\right), \tag{4.41}$$

where $K=max\{2+\|u_x\|_\infty,1,1+\|u_x\|_\infty,\|\eta\|_\infty^2+1\}$ *is bounded, as a consequence of (4.36).*

Then, applying the Gronwall's lemma in (4.41) we conclude that

$$\max_{0\leq t\leq T}(\|V(t)\|_1+\|S(t)\|_1)\leq C(T,b,d,\|V_0\|_1,\|S_0\|_1). \tag{4.42}$$

This fact ensures that the solution cannot blow-up, so we can extend the local solution to a solution on every bounded interval $[0,T]$.

Finally, we are ready to prove the following theorem, corresponding to the second part of this analysis.

Theorem 15. *Let $(\eta, u) \in C(0,T; H^s(\Omega) \times H^s(\Omega))$ be the solution of (4.24) corresponding to initial data $(\eta_0, u_0) \in H^s(\Omega) \times H^s(\Omega)$, $s \geq 2$. Then, there exists a unique solution $(\eta_N, u_N) \in S_N \times S_N$ of (4.25) for all finite time $T > 0$. Moreover, given $0 \leq t \leq T$, there exists a constant $C = C(T, b, d) > 0$ independent of N such that,*

$$\max_{0\leq t\leq T} (\|\eta(t) - \eta_N(t)\|_1 + \|u(t) - u_N(t)\|_1) \leq CN^{1-s}. \tag{4.43}$$

Proof. The existence and uniqueness can be proved through the theory of ODE systems, similarly to what was done in remark 5.

Let $e_1 = V - \eta_N$ and $e_2 = S - u_N$. Then, we can write

$$\begin{aligned} \eta - \eta_N = \eta - (V - e_1) = (\eta - V) + e_1, \\ u - u_N = u - (S - e_2) = (u - S) + e_2. \end{aligned} \tag{4.44}$$

As a consequence of (4.27), to obtain (4.43), we only have to estimate e_1 and e_2. With this aim in mind, we subtract (4.26) and (4.25) and we get

$$\begin{aligned} (e_{1_t}, \varphi) + (e_{2_x}, \varphi) + ((uV_x + \eta S_x - (\eta_N u_N)_x), \varphi) - b(e_{1_{xxt}}, \varphi) = 0, \\ (e_{2_t}, \psi) + (e_{1_x}, \psi) + (uS_x - u_N u_{N_x}, \psi) - d(e_{2_{xxt}}, \psi) = 0. \end{aligned} \tag{4.45}$$

If we note that

$$uV_x + \eta S_x - (\eta_N u_N)_x = V_x(u - S) + S_x(\eta - V) + (Se_1)_x + (Ve_2)_x - (e_1 e_2)_x, \tag{4.46}$$

and

$$uS_x - u_N u_{N_x} = (u - S)S_x + (Se_2)_x - e_2 e_{2_x}, \tag{4.47}$$

we can use the definition of e_1 and e_2, and proceeding similarly to the demonstration of Lemma 3, taking $\varphi = e_1$ and $\psi = e_2$ in (4.45), we obtain the following system

$$\begin{aligned} a_B(e_{1_t}, e_1) + (e_{2_x} + V_x(u - S) + S_x(\eta - V) + (Se_1)_x + (Ve_2)_x - (e_1 e_2)_x, e_1) = 0, \\ a_D(e_{2_t}, e_2) + (e_{1_x}, e_2) + ((u - S)S_x + (Se_2)_x - e_2 e_{2_x}, e_2) = 0, \end{aligned} \tag{4.48}$$

with $e_1(0) = e_2(0) = 0$.

Let $t_h \in (0, T]$ be the maximal temporal instance for which (4.48) has a unique solution such that

$$\|e_1(t)\|_\infty \leq 1, \quad \text{for} \quad 0 \leq t \leq t_h. \tag{4.49}$$

Proceeding as before, using (4.49), the proposition 5, the Cauchy-Schwarz and Young inequalities in (4.48), and adding both resulting equations, we obtain that for $0 \leq t \leq t_h$ it holds

$$\frac{d}{dt}\left(\|e_1\|_1^2 + \|e_2\|_1^2\right) \leq c_1 + c_2\left(\|e_1\|_1^2 + \|e_2\|_1^2\right), \tag{4.50}$$

where $c_1 = \|V_x\|_\infty^2 \|u - S\|^2 + \|S_x\|_\infty^2 \|\eta - V\|^2 + \|S_x\|_\infty^2 \|u - S\|^2$ and $c_2 = \max\{5 + 3\|S_x\|_\infty, 1, 3 + 3\|S_x\|_\infty + \|V_x\|_\infty^2, 3 + \|V\|_\infty^2\}$.

Note that, using Sobolev's lemma, (4.5), (4.6), the inverse estimates for elements of S_N given by (4.7), and (4.37), we have

$$\begin{aligned} \|V_x\|_\infty &\leq \|\eta_x\|_\infty + \|(\eta - P_N\eta)_x\|_\infty + \|(P_N\eta - V)_x\|_\infty \\ &\leq C + CN^{2-s} + CN^{\frac{3}{2}-s}, \end{aligned} \tag{4.51}$$

$$\begin{aligned} \|S_x\|_\infty &\leq \|u_x\|_\infty + \|(u - P_Nu)_x\|_\infty + \|(P_Nu - S)_x\|_\infty \\ &\leq C + CN^{2-s} + CN^{\frac{3}{2}-s}, \end{aligned} \tag{4.52}$$

$$\begin{aligned} \|V\|_\infty &\leq \|\eta\|_\infty + \|\eta - P_N\eta\|_\infty + \|P_N\eta - V\|_\infty \\ &\leq C + CN^{1-s} + CN^{1-s}, \end{aligned} \tag{4.53}$$

which are all bounded by a constant C, since $s \geq 2$.

The estimates given by (4.51), (4.52) and (4.53) along Lemma 3, lead us to conclude that $c_1 \leq CN^{2(1-s)}$ and that c_2 is bounded. Then, integrating (4.50) over $[0, t]$ and using the Gronwall's lemma, we conclude that

$$\|e_1(t)\|_1 + \|e_2(t)\|_1 \leq C(T, b, d)N^{1-s}, \tag{4.54}$$

that holds for all $t \in [0, t_h]$. To extend this result to $[0, T]$, observe that by the Sobolev lemma we have

$$\|e_1(t)\|_\infty \leq C\|e_1(t)\|_1 \leq CN^{1-s},$$

for $0 \leq t \leq t_h$, if we consider the limitation given by (4.54).

However, since $s \geq 2$, N^{1-s} goes to zero when N approaches to infinity for all t. Therefore, $\|e_1(t)\|_\infty \leq 1$ for all t. Therefore, we can take $t_h = T$ in (4.49). The estimate (4.43) follows by (4.54) and Lemma 3. □

4.3.1.2 Weakly Dispersive Systems Case I - H has order 0

For this case, we assume that $b, d > 0$ and $a = c > 0$ or $a, c < 0$. Then, the Boussinesq system becomes:

$$\begin{aligned} \eta_t + u_x + (\eta u)_x + au_{xxx} - b\eta_{xxt} = 0, \\ u_t + \eta_x + uu_x + c\eta_{xxx} - du_{xxt} = 0, \end{aligned} \tag{4.55}$$

with initial conditions $\eta(x, 0) = \eta_0(x)$ and $u(x, 0) = u_0(x)$, for $x \in \Omega$ and $t \geq 0$.

In order to obtain error bounds for the semi discrete problem obtained from (4.55) using the Fourier collocation method, we consider weight functions as functions from S_N as well, and require that the following system is satisfied

$$\begin{aligned} (\eta_{N_t}, \varphi) + (u_{N_x}, \varphi) + ((\eta_N u_N)_x, \varphi) + a(u_{N_{xxx}}, \varphi) - b(\eta_{N_{xxt}}, \varphi) = 0, \\ (u_{N_t}, \psi) + (\eta_{N_x}, \psi) + (u_N u_{N_x}, \psi) + c(\eta_{N_{xxx}}, \psi) - d(u_{N_{xxt}}, \psi) = 0, \end{aligned} \tag{4.56}$$

in Ω and for $t \geq 0$, where $\varphi, \psi \in S_N$ are the weight functions. The system (4.56) is supplemented with the initial conditions $\eta_N(x, 0) = P_N\eta_0(x)$ and $u_N(x, 0) = P_N u_0(x)$, for $x \in \Omega$.

Moreover, as it was seen on the Section 3.3.2, the inital condition and the solution in this case satisfy that $(\eta_0, u_0) \in H^s(\Omega) \times H^s(\Omega)$ and $(\eta, u) \in C(0, T; H^s(\Omega) \times$

$H^s(\Omega))$, for $s \geq 0$ and $T > 0$.

We proceed similarly to what was done in Section 4.3.1.1. For the first part, we seek for functions $V, S \in S_N$ such that, given $(\eta, u) \in C(0, T; H^s(\Omega)^2)$ solution of (4.55) corresponding to initial data $(\eta_0, u_0) \in (H^s(\Omega))^2$, satisfy for all $\varphi, \psi \in S_N$, the following system:

$$\begin{aligned} (V_t, \varphi) + (S_x, \varphi) + ((uV_x + \eta S_x), \varphi) + a(S_{xxx}, \varphi) - b(V_{xxt}, \varphi) = 0, \\ (S_t, \psi) + (V_x, \psi) + (uS_x, \psi) + c(V_{xxx}, \psi) - d(S_{xxt}, \psi) = 0, \end{aligned} \tag{4.57}$$

with initial conditions given by $V(x, 0) = P_N\eta_0(x)$ and $S(x, 0) = P_N u_0(x)$ for $x \in \Omega$.

We have the following result:

Lemma 4. *Let $s \geq 2$ and $(\eta, u) \in C(0, T; H^s(\Omega) \times H^s(\Omega))$ the solution of (4.55) corresponding to initial data $(\eta_0, u_0) \in H^s(\Omega) \times H^s(\Omega)$, $s \geq 2$. Then, there is a unique solution $(V, S) \in S_N \times S_N$ of (4.57) for all finite time $T \geq 0$. Moreover, given $0 \leq t \leq T$, there is a constant $C = C(T, a, b, c, d) > 0$ independent of N, such that*

$$\max_{0 \leq t \leq T} \{\|\eta(t) - V\|_1 + \|u(t) - S\|_1\} \leq CN^{1-s}. \tag{4.58}$$

Proof. We consider $\rho = P_N\eta - V$ and $\theta = P_N u - S$, so that (4.28) and (4.5) hold.

As it was done in Lemma 3, we use (4.28), (4.30) and (4.31) on the difference between

$$\begin{aligned} (\eta_t, \varphi) + (u_x, \varphi) + ((\eta u)_x, \varphi) + a(u_{xxx}, \varphi) - b(\eta_{xxt}, \varphi) = 0, \\ (u_t, \psi) + (\eta_x, \psi) + (uu_x, \psi) + c(\eta_{xxx}, \psi) - d(u_{xxt}, \psi) = 0, \end{aligned}$$

and (4.57), to obtain the following ODE system

$$\begin{aligned} a_B(\rho_t, \varphi) + (\theta_x + u(\eta - P_N\eta)_x + u\rho_x + \eta(u - P_N u)_x + \eta\theta_x, \varphi) + a(\theta_{xxx}, \varphi) = 0, \\ a_D(\theta_t, \psi) + (\rho_x, \psi) + (u(u - P_N u)_x + u\theta_x, \psi) + c(\rho_{xxx}, \psi = 0, \end{aligned} \tag{4.59}$$

with initial conditions $\rho(0) = \theta(0) = 0$.

If we take $\varphi = \rho$ and $\psi = \theta$ in (4.59), then

$$\begin{aligned} a_B(\rho_t, \rho) = &-(\theta_x, \rho) - (u(\eta - P_N\eta)_x, \rho) - (u\rho_x, \rho) - (\eta(u - P_N u)_x, \rho) \\ &- (\eta\theta_x, \rho) + a(\theta_{xx}, \rho_x), \end{aligned} \tag{4.60}$$

and

$$a_D(\theta_t, \theta) = -(\rho_x, \theta) - (u(u - P_N u)_x, \theta) - (u\theta_x, \theta) - c(\rho_x, \theta_{xx}). \tag{4.61}$$

Multiplying (4.60) by $|c|$ and (4.61) by $|a|$, adding both equation and using the Cauchy-Schwarz and Young inequalities and the characteristics of a_B, a_D, we get that

$$\frac{d}{dt}\left(\|\rho\|_1^2 + \|\theta\|_1^2\right) \leq c_3 + c_4\left(\|\rho\|_1^2 + \|\theta\|_1^2\right), \tag{4.62}$$

where

$$\begin{aligned} c_3 = &(\min\{|c|, |a|\})^{-1}|c|\left(\|u\|_\infty^2\|(\eta - P_N\eta)_x\|^2 + \|\eta\|_\infty^2\|(u - P_N u)_x\|^2\right) + \\ &|a|\|u\|_\infty^2\|(u - P_N u)_x\|^2, \end{aligned}$$

and $c_4 = (\min\{|c|, |a|\})^{-1}\max\{4|c| + |c|\|u_x\|_\infty, |a|, 2|a| + |a|\|u_x\|_\infty, |c| + |c|\|\eta\|_\infty^2\}$.

Remark 6. *To obtain (4.62) we just observe that, when a and c are equal or have the same signal, i.e., both are positives or negative, we have $|c|a(\theta_{xx}, \rho_x) = |a|c(\rho_x, \theta_{xx})$.*

The Sobolev lemma along the result about the exact solution and Proposition 4, lead us to conclude that $c_3 \leq CN^{2(1-s)}$ and c_4 is bounded for $s \geq 2$.

Thus, integrating (4.62) over $[0, t]$ and applying the Gronwall's lemma we get

$$\|\rho(t)\|_1 + \|\theta(t)\|_1 \leq C(T, a, b, c, d)N^{1-s}. \tag{4.63}$$

The estimates (4.63) and (4.5) imply (4.58).

As in Lemma 3, to finish this proof, we have to demonstrate that (4.57) has a unique solution for all $t \geq 0$. The existence and uniqueness of a local solution can be proved through the theory of ODE systems, like was done in remark 5. Its extension for all $[0, T]$ is possible because if we take $\varphi = V$ and $\psi = S$ in (4.57), it is possible to prove that

$$\max_{0 \leq t \leq T} (\|V(t)\|_1 + \|S(t)\|_1) \leq C(T, a, b, c, d, \|V_0\|_1, \|S_0\|_1), \tag{4.64}$$

which ensures that the solution cannot blow up. □

The next step is to prove a estimative between the solution (η, u) of (4.55) and (η_N, u_N) of (4.56). This result can be found in the following Theorem.

Theorem 16. *Let $(\eta, u) \in C(0, T; H^s(\Omega) \times H^s(\Omega))$ be solution of (4.55) corresponding to the initial data $(\eta_0, u_0) \in H^s(\Omega) \times H^s(\Omega)$, for $s \geq 2$. Then, there exists a time $T > 0$ and a constant $C = C(T, a, b, c, d) > 0$, independent of N, such that,*

$$\max_{0 \leq t \leq T} (\|\eta(t) - \eta_N\|_1 + \|u(t) - u_N\|_1) \leq CN^{1-s}. \tag{4.65}$$

Proof. The existence and uniqueness of solution of (4.56) is guaranteed for the theory of ODE systems, like in Theorem 15.

Let $e_1 = V - \eta_N$ and $e_2 = S - u_N$, as in Theorem 15. Once again, to conclude (4.65), we only need to estimate e_1 and e_2. With this aim in mind, we subtract (4.57) and (4.56), use the decomposition given in (4.44), (4.46) and (4.47) and take $\varphi = e_1$ and $\psi = e_2$, to obtain the following system

$$\begin{aligned} a_B(e_{1_t}, e_1) + (e_{2_x} + V_x(u - S) + S_x(\eta - V) + (Se_1)_x + (Ve_2)_x - (e_1 e_2)_x, e_1) & \\ + a(e_{2_{xxx}}, e_1) = 0, & \quad (4.66) \\ a_D(e_{2_t}, e_2) + (e_{1_x}, e_2) + ((u - S)S_x + (Se_2)_x - e_2 e_{2_x}, e_2) + c(e_{1_{xxx}}, e_2) = 0, & \end{aligned}$$

with $e_1(0) = e_2(0) = 0$.

Let $t_h \in (0, T]$ be the maximal temporal instance for which (4.66) has a unique solution such that (4.49) holds.

Similarly to the demonstration of the Lemma 4, considering the Remark 6, using the Cauchy-Schwarz and Young inequalities, we get that

$$\frac{d}{dt}\left(\|e_1\|_1^2 + \|e_2\|_1^2\right) \le c_7 + c_8\left(\|e_1\|_1^2 + \|e_2\|_1^2\right), \tag{4.67}$$

where

$$c_7 = (\min\{|c|, |a|\})^{-1}\left(|c|\|V_x\|_\infty^2\|u - S\|^2 + |c|\|S_x\|_\infty^2\|\eta - V\|^2 + |a|\|S_x\|_\infty^2\|u - S\|^2\right)$$

and

$$\begin{aligned} c_8 =& (\min\{|c|, |a|\})^{-1}\max\left\{5|c| + 3|c|\|S_x\|_\infty, |a|, |c| + 2|a| + 3|a|\|S_x\|_\infty + |c|\|V_x\|_\infty^2,\right. \\ &\left. 3|c| + |c|\|V\|_\infty^2\right\}. \end{aligned}$$

The estimates (4.51), (4.52), (4.53) and Lemma 4 lead us to conclude that c_8 is bounded and that $c_7 \le C(a, b, c, d)N^{2(1-s)}$. Then, integrating (4.67) over $[0, t]$ and using the Gronwall's lemma, we conclude that

$$\|e_1(t)\|_1 + \|e_2(t)\|_1 \le C(T, a, b, c, d)N^{1-s}, \tag{4.68}$$

that holds for all $t \in [0, t_h]$.

To extend this result to $[0, T]$, we proceed as in Theorem 15. Therefore, the Theorem follows by (4.68) and the result obtained in Lemma 4. $\square$

4.3.1.3 Weakly Dispersive Systems Case II - H has order −1

For this case, we assume that $b, d > 0$, $a = 0$ and $c < 0$. Then, the Boussinesq system becomes:

$$\begin{aligned} \eta_t + u_x + (\eta u)_x - b\eta_{xxt} = 0, \\ u_t + \eta_x + uu_x + c\eta_{xxx} - du_{xxt} = 0, \end{aligned} \tag{4.69}$$

with initial conditions $\eta(x, 0) = \eta_0(x)$ and $u(x, 0) = u_0(x)$, for $x \in \Omega$ and $t \geq 0$.

In order to obtain error bounds for the semi discrete problem obtained from (4.69) using the Fourier collocation method, we consider weight functions φ, ψ as functions from S_N as well, and require that the following system is satisfied

$$\begin{aligned} (\eta_{N_t}, \varphi) + (u_{N_x}, \varphi) + ((\eta_N u_N)_x, \varphi) - b(\eta_{N_{xxt}}, \varphi) = 0, \\ (u_{N_t}, \psi) + (\eta_{N_x}, \psi) + (u_N u_{N_x}, \psi) + c(\eta_{N_{xxx}}, \psi) - d(u_{N_{xxt}}, \psi) = 0, \end{aligned} \tag{4.70}$$

in Ω and for $t \geq 0$. The system (4.70) is supplemented with the initial conditions $\eta_N(x, 0) = P_N\eta_0(x)$ and $u_N(x, 0) = P_N u_0(x)$, for $x \in \Omega$.

Moreover, as it was seen on the Section 3.3.2, the inital condition and the solution in this case satisfy that $(\eta_0, u_0) \in H^{s+1}(\Omega) \times H^s(\Omega)$ and $(\eta, u) \in C(0, T; H^{s+1}(\Omega) \times H^s(\Omega))$ for $s \geq 0$ and $T > 0$.

The analysis is done in two parts, like in the previous sections. For the first part, given $(\eta, u) \in C(0, T; H^{s+1}(\Omega) \times H^s(\Omega))$ solution of (4.69) corresponding to initial data $(\eta_0, u_0) \in H^{s+1}(\Omega) \times H^s(\Omega)$, we seek for functions $V, S \in S_N$ satisfying the following linearized system for all $\varphi, \psi \in S_N$

$$\begin{aligned} (V_t, \varphi) + (S_x, \varphi) + ((uV_x + \eta S_x), \varphi) - b(V_{xxt}, \varphi) = 0, \\ (S_t, \psi) + (V_x, \psi) + (uS_x, \psi) + c(V_{xxx}, \psi) - d(S_{xxt}, \psi) = 0, \end{aligned} \tag{4.71}$$

with initial conditions given by $V(x, 0) = P_N\eta_0(x)$ and $S(x, 0) = P_N u_0(x)$ for $x \in \Omega$.

We have the following result:

Lemma 5. *Let $(\eta, u) \in C(0,T; H^{s+1}(\Omega) \times H^s(\Omega))$ be the solution of (4.69) corresponding to initial data $(\eta_0, u_0) \in H^{s+1}(\Omega) \times H^s(\Omega)$ for $s \geq 2$. Then, there is a unique solution $(V, S) \in S_N$ of (4.71) for all finite time $T \geq 0$. Moreover, given $0 \leq t \leq T$, there is a constant $C = C(T, b, c, d) > 0$ independent of N, such that*

$$\max_{0 \leq t \leq T} \{\|\eta(t) - V\|_2 + \|u(t) - S\|_1\} \leq CN^{2-s}. \tag{4.72}$$

Proof. First of all, we observe that the L^2-orthogonal projection onto S_N represented by P_N is stable in $H^2_{\text{per}}(\Omega)$. Indeed,

$$\|P_N v\|_{H^2_{\text{per}}} = \|P_N v - v + v\|_2 \leq \|P_N v - v\|_2 + \|v\|_2 \leq CN^{2-2}\|v\|_2 + \|v\|_2,$$

for all $v \in H^s(\Omega)$ and $s \geq 2$. Then, it holds that

$$\|P_N v\|_2 \leq C\|v\|_2. \tag{4.73}$$

Second of all, we have that

$$\begin{aligned} \|f\|^2 = \int_{-\pi}^{\pi} (\Upsilon - b\Upsilon_{xx})^2 \, dx &= \int_{-\pi}^{\pi} \Upsilon^2 dx + 2b \int_{-\pi}^{\pi} (\Upsilon_x)^2 dx + b^2 \int_{-\pi}^{\pi} (\Upsilon_{xx})^2 dx \\ &\geq \min\left\{1, 2b, b^2\right\} \|\Upsilon\|_2^2. \end{aligned} \tag{4.74}$$

Let $\rho = P_N\eta - V$ and $\theta = P_N u - S$, so that (4.28) and (4.5) hold. As in the previous sections, if we use the decomposition given in (4.28) on the difference between

$$\begin{aligned} (\eta_t, \varphi) + (u_x, \varphi) + ((\eta u)_x, \varphi) - b(\eta_{xxt}, \varphi) &= 0, \\ (u_t, \psi) + (\eta_x, \psi) + (uu_x, \psi) + c(\eta_{xxx}, \psi) - d(u_{xxt}, \psi) &= 0, \end{aligned}$$

and (4.71), we get

$$\begin{aligned} (\rho_t, \varphi) + (\theta_x, \varphi) + (u(\eta - P_N\eta)_x + u\rho_x + \eta(u - P_N u)_x + \eta\theta_x, \varphi) - b(\rho_{xxt}, \varphi) &= 0, \\ (\theta_t, \psi) + (\rho_x, \psi) + (u(u - P_N u)_x + u\theta_x, \psi) + c(\rho_{xxx}, \psi) - d(\theta_{xxt}, \psi) &= 0. \end{aligned} \tag{4.75}$$

The first equation of (4.75) can be written as $(\rho_t - b\rho_{t_{xx}}, \varphi) = (g(x), \varphi)$, where $g(x) = -(\theta_x + u(\eta - P_N\eta)_x + u\rho_x + \eta(u - P_N u)_x + \eta\theta_x)$. If we consider $\rho_t = P_N\xi$, where ξ is solution of the problem $\xi - b\xi_{xx} = g(x)$ in Ω with periodic boundary conditions, using (4.73) and (4.74), we can show that

$$\|\rho_t\|_2^2 = \|P_N\xi\|_2^2 \leq C\|\xi\|_2^2 \leq C\|g\|^2.$$

Therefore, the fact that $H^2(\Omega) \subset H^1(\Omega)$ and the estimate given by Lemma 4, lead us to conclude that

$$\|\rho_t\|_2^2 \leq k_3\|\rho_x\|^2 + k_4\|\theta_x\|^2 + CN^{2(2-s)}, \tag{4.76}$$

where $k_3 = 1 + 2\|u\|_\infty^2 + 2\|\eta u\|_\infty^2$ and $k_4 = 2 + 2\|u\|_\infty^2 + 3\|\eta\|_\infty^2 + \|\eta u\|_\infty^2$ are bounded as a consequence of (4.36).

On the other hand, if we choose $\psi = \theta$ in the second equation of (4.75) and use again Proposition 4, we get that

$$\|\theta_t\|_1^2 \leq C\|\rho\|^2 + C\|\theta\|^2 + C\|\rho_{xx}\|^2 + C\|\theta_x\|^2 + CN^{2(2-s)}. \tag{4.77}$$

Adding (4.76) and (4.77), we have

$$\|\rho_t\|_2^2 + \|\theta_t\|_1^2 \leq CN^{2(2-s)} + C\left(\|\rho\|_2^2 + \|\theta\|_1^2\right),$$

and after integrating it over $[0, t]$, the Gronwall's lemma lead us to conclude that

$$\|\rho(t)\|_2 + \|\theta(t)\|_1 \leq C(T, b, c, d)N^{2-s}. \tag{4.78}$$

To finish this proof, we observe that the existence and uniqueness of solution at least local for the problem (4.71) is guaranteed by the systems of ODE theory.

Its extension to the entire $[0, T]$ can be seen as follows: the first equation of (4.71) can be written as:

$$(V_t - bV_{t_{xx}}, \varphi) = -(S_x + uV_x + \eta S_x, \varphi) \tag{4.79}$$

Similarly to before, considering $V_t = P_N \Upsilon$ where Υ is the solution of the problem $\Upsilon - b\Upsilon_{xx} = f(x)$ in Ω with periodic boundary conditions and $f(x) = -(S_x + uV_x + \eta S_x)$, using (4.73) and (4.74), we can show that

$$\|V_t\|_2^2 \;\leq\; C\|f\|^2 \leq k_1\|S_x\|^2 + k_2\|V_x\|^2, \tag{4.80}$$

where $k_1 = 1 + 2\|\eta\|_\infty + \|u\|_\infty^2 + \|\eta\|_\infty^2 + \|\eta u\|_\infty^2$ and $k_2 = 2 + \|u\|_\infty^2$ are bounded as a consequence of (4.36).

On the other hand, if we take $\psi = S$ in the second equation of (4.71) we have that

$$\|S_t\|_1^2 \leq -2(V_x, S) + 2(u_x, S^2) + 2c(V_{xx}, S_x). \tag{4.81}$$

Using the Cauchy Schwarz and Young inequalities in (4.81) and adding it with (4.80) we obtain that

$$\|V_t\|_2^2 + \|S_t\|_1^2 \leq C(b, c, d)\left(\|V\|_2^2 + \|S\|_1^2\right). \tag{4.82}$$

After integrating (4.82) over $[0, t]$, we apply the Gronwall's lemma to conclude that

$$\|V(t)\|_2 + \|S(t)\|_1 \leq C(T, b, c, d, \|V_0\|_2, \|S_0\|_1).$$

This fact ensures that the solution of (4.71) cannot blow-up and can be extend over the entire interval $[0, T]$. $\square$

As in the previous sections, we are ready to prove the following result.

Theorem 17. *Let* $(\eta, u) \in C(0,T; H^{s+1}(\Omega) \times H^s(\Omega))$ *be the solution of (4.69) corresponding to the initial data* $(\eta_0, u_0) \in H^{s+1}(\Omega) \times H^s(\Omega)$ *for* $s \geq 2$. *Then, there exists a finite time* $T > 0$ *and a constant* $C = C(T, a, b, c, d) > 0$ *independent of* N, *such that*

$$\max_{0 \leq t \leq T} (\|\eta(t) - \eta_N\|_2 + \|u(t) - u_N\|_1) \leq CN^{2-s}. \tag{4.83}$$

Proof. The existence and uniqueness of solution of (4.70) is guaranteed for the theory of ODE systems, like in Theorems 15 and 16 .

Once again, let $e_1 = V - \eta_N$ and $e_2 = S - u_N$. Using the same arguments of Theorems 15 and 16, we obtain the following system:

$$\begin{aligned} a_B(e_{1_t}, e_1) + (e_{2_x} + V_x(u - S) + S_x(\eta - V) + (Se_1)_x + (Ve_2)_x - (e_1e_2)_x, e_1) = 0, \\ a_D(e_{2_t}, e_2) + (e_{1_x}, e_2) + ((u - S)S_x + (Se_2)_x - e_2e_{2_x}, e_2) + c(e_{1_{xxx}}, e_2) = 0, \end{aligned} \tag{4.84}$$

with $e_1(0) = e_2(0) = 0$.

Observe that the first equation of (4.84) imply that

$$(e_{1_t} - be_{1_{xxt}}, e_1) = -(e_{2_x} + V_x(u - S) + S_x(\eta - V) + (Se_1)_x + (Ve_2)_x - (e_1e_2)_x, e_1).$$

Taking $e_{1_t} = P_N\Theta$, where Θ is solution of $\Theta - b\Theta_{xx} = h(x)$ in Ω with periodic boundary conditions and $h(x) = -(e_{2_x} + V_x(u - S) + S_x(\eta - V) + (Se_1)_x + (Ve_2)_x - (e_1e_2)_x)$, we can use the definition of e_{1_t} and (4.73) and (4.74) to show that

$$\|e_{1_t}\|_2^2 = \|P_N\Theta\|_2^2 \leq C\|\Theta\|_2^2 \leq C\|h\|^2.$$

Let $t_h \in (0, T]$ be the maximal temporal instance for which (4.84) has a unique solution such that (4.49) holds.

The arguments used to estimate $\|f\|^2$ and $\|g\|^2$ in Lemma 5 also provide an estimate for $\|h\|^2$, and lead us to conclude that

$$\|e_{1_t}\|_2 \leq C\left(\|e_1\|_1 + \|e_2\|_1\right) + CN^{2-s}. \tag{4.85}$$

On the other hand, applying the Cauchy Schwarz and Young inequalities on the second equation of (4.84), we have that

$$\|e_{2_t}\|_1 \leq C\|S_x\|_\infty\|u - S\| + C\|S_x\|_\infty\|e_2\| + c\|e_{1_{xx}}\| + c\|e_{2_x}\|. \tag{4.86}$$

Adding (4.85) and (4.86), and using the Lemma 5 along with (4.51), (4.52) and (4.53), we have that

$$\|e_{1_t}\|_2 + \|e_{2_t}\|_1 \leq CN^{2-s} + C\left(\|e_1\|_2 + \|e_2\|_1\right).$$

Integrating this inequality over $[0, t]$ and using the Gronwall's lemma, we obtain that

$$\|e_1(t)\|_2 + \|e_2(t)\|_1 \leq C(T, b, c, d)N^{2-s}, \tag{4.87}$$

which holds for all $t \in [0, t_h]$.

To extend this result to $[0, T]$, we resort to the similar argument used in Theorem 15. The estimate (4.83) follows by (4.87) and Lemma 5. □

4.3.1.4 Weakly Dispersive Systems Case III - H has order 1

For this case, we assume that $b, d > 0$, $c = 0$ and $a < 0$. Then, the Boussinesq system becomes:

$$\begin{aligned} \eta_t + u_x + (\eta u)_x + au_{xxx} - b\eta_{xxt} &= 0, \\ u_t + \eta_x + uu_x - du_{xxt} &= 0, \end{aligned} \tag{4.88}$$

with initial conditions $\eta(x,0) = \eta_0(x)$ and $u(x,0) = u_0(x)$, for $x \in \Omega$ and $t \geq 0$.

In order to obtain error bounds for the semi discrete problem obtained from (4.69) using the Fourier collocation method, we consider weight functions φ, ψ as functions from S_N as well, and require that the following system is satisfied

$$\begin{aligned} (\eta_{N_t}, \varphi) + (u_{N_x}, \varphi) + ((\eta_N u_N)_x, \varphi) + a(u_{N_{xxx}}, \varphi) - b(\eta_{N_{xxt}}, \varphi) = 0, \\ (u_{N_t}, \psi) + (\eta_{N_x}, \psi) + (u_N u_{N_x}, \psi) - d(u_{N_{xxt}}, \psi) = 0, \end{aligned} \tag{4.89}$$

in Ω and for $t \geq 0$. The system (4.89) is supplemented with the initial conditions $\eta_N(x,0) = P_N \eta_0(x)$ and $u_N(x,0) = P_N u_0(x)$, for $x \in \Omega$.

Moreover, as it was seen on the Section 3.3.2, the inital condition and the solution in this case satisfy that $(\eta_0, u_0) \in H^s(\Omega) \times H^{s+1}(\Omega)$ and $(\eta, u) \in C(0,T; H^s(\Omega) \times H^{s+1}(\Omega))$ for $s \geq 0$ and $T > 0$.

Following the idea of dividing the analysis in two parts, we have the following results

Lemma 6. *Let $(\eta, u) \in C(0,T; H^s(\Omega) \times H^{s+1}(\Omega))$ be the solution of (4.88) corresponding to initial data $(\eta_0, u_0) \in H^s(\Omega) \times H^{s+1}(\Omega)$ for $s \geq 2$. Then, there is a unique solution $(V,S) \in S_N \times S_N$ of*

$$\begin{aligned} (V_t, \varphi) + (S_x, \varphi) + ((uV_x + \eta S_x), \varphi) + a(S_{xxx}, \varphi) - b(V_{xxt}, \varphi) = 0, \\ (S_t, \psi) + (V_x, \psi) + (uS_x, \psi) - d(S_{xxt}, \psi) = 0, \end{aligned}$$

for all time $T \geq 0$. Moreover, given $0 \leq t \leq T$, there is a constant $C = C(T,a,b,d) > 0$, independent of N such that,

$$\max_{0 \leq t \leq T} \{\|\eta(t) - V\|_1 + \|u(t) - S\|_2\} \leq CN^{2-s}. \tag{4.90}$$

Theorem 18. *Let $(\eta, u) \in C(0,T; H^s(\Omega) \times H^{s+1}(\Omega))$ be the solution of (4.88) corresponding to the initial data $(\eta_0, u_0) \in H^s(\Omega) \times H^{s+1}(\Omega)$ for $s \geq 2$. Then, there exists a time $T > 0$ and a constant $C = C(T,a,b,c,d) > 0$ independent of N, such that*

$$\max_{0 \leq t \leq T} (\|\eta(t) - \eta_N\|_1 + \|u(t) - u_N\|_2) \leq CN^{2-s}. \tag{4.91}$$

The idea to prove these results is analogous to the one used on sub section 4.3.1.3, with respect to Case II. The roles of V and S and ρ and θ are reversed, during the demonstration of the Lemma 6, as well as, the roles of e_1 and e_2, during the demonstration of Theorem 18.

4.3.1.5 Purely KdV-type Boussinesq System

For this case, we assume that $b, d = 0$ and $a = c = \frac{1}{6}$. Then, the Boussinesq system becomes:

$$\begin{aligned} \eta_t + u_x + (\eta u)_x + \tfrac{1}{6} u_{xxx} &= 0, \\ u_t + \eta_x + u u_x + \tfrac{1}{6} \eta_{xxx} &= 0, \end{aligned} \tag{4.92}$$

with initial conditions $\eta(x,0) = \eta_0(x)$ and $u(x,0) = u_0(x)$ for $x \in \Omega$ and $t \geq 0$.

In order to obtain error bounds for the semi discrete problem obtained from (4.92) using the Fourier collocation method, we consider weight functions φ, ψ as functions from S_N as well, and require that the following system is satisfied

$$\begin{aligned} (\eta_{N_t}, \varphi) + (u_{N_x}, \varphi) + ((\eta_N u_N)_x, \varphi) + \tfrac{1}{6}(u_{N_{xxx}}, \varphi) &= 0, \\ (u_{N_t}, \psi) + (\eta_{N_x}, \psi) + (u_N u_{N_x}, \psi) + \tfrac{1}{6}(\eta_{N_{xxx}}, \psi) &= 0, \end{aligned} \tag{4.93}$$

in Ω and for $t \geq 0$. The system (4.93) is supplemented with the initial conditions $\eta_N(x,0) = P_N \eta_0(x)$ and $u_N(x,0) = P_N u_0(x)$, for $x \in \Omega$.

Moreover, as it was seen on the Section 3.3.2, the inital condition and the solution in this case satisfy that $(\eta_0, u_0) \in H^s(\Omega) \times H^s(\Omega)$ and $(\eta, u) \in C(0, T; H^s(\Omega) \times H^s(\Omega))$ for $s > \frac{3}{4}$ and $T > 0$.

Following the idea of dividing the analysis in two parts as in the previous sections, we have the following results

Lemma 7. *Let $(\eta, u) \in C(0, T; H^s(\Omega) \times H^s(\Omega))$ be the solution of (4.92) corresponding to initial data $(\eta_0, u_0) \in H^s(\Omega) \times H^s(\Omega)$ for $s \geq 2$. Then, there is a unique* *solution* $(V,S) \in S_N \times S_N$ *of*

$$\begin{aligned} (V_t,\varphi)+(S_x,\varphi)+((\eta_x S+u_x V),\varphi)+\tfrac{1}{6}(S_{xxx},\varphi)&=0,\\ (S_t,\psi)+(V_x,\psi)+(u_x S,\psi)+\tfrac{1}{6}(V_{xxx},\psi)&=0, \end{aligned} \tag{4.94}$$

for all finite time $T \geq 0$. *Moreover, given* $0 \leq t \leq T$, *there is a constant* $C = C(T) > 0$ *independent of* N, *such that*

$$\max_{0\leq t\leq T} \{\|\eta(t)-V\|+\|u(t)-S\|\} \leq CN^{-s}. \tag{4.95}$$

Proof. The ODE system theory implies that (4.94) has a unique local solution in $[0,t_h]$, $t_h < T$. To extend this solution to $[0,T]$, we take $\varphi = V$ and $\psi = S$ in (4.94) to obtain from the first equation that

$$\frac{1}{2}\frac{d}{dt}\|V\|^2 = -(S_x,V)-(\eta_x S+u_x V,V)-\frac{1}{6}(S_{xxx},V), \tag{4.96}$$

and from the second equation that

$$\frac{1}{2}\frac{d}{dt}\|S\|^2 = -(V_x,S)-(u_x S,S)-\frac{1}{6}(V_{xxx},S). \tag{4.97}$$

Adding (4.96) and (4.97) and observing that the first and the third terms on the right side of these equations are equal if we integrate by parts, we get that

$$\frac{d}{dt}\left(\|V\|^2+\|S\|^2\right) \leq c_1\left(\|V\|^2+\|S\|^2\right), \tag{4.98}$$

where $c_1 = \max\ \{1+2\|u_x\|_\infty, \|\eta_x\|_\infty^2+2\|u_x\|_\infty\}$ is a bounded constant, as a consequence of (4.36).

Integrating (4.98) over $[0,t]$ and using the Gronwall's lemma, we conclude that

$$\max_{0\leq t\leq T} \|V\|+\|S\| \leq C(T,\|V_0\|,\|S_0\|),$$

which implies that the solution of (4.94) cannot blow up and we can extend the solution over $[0,T]$.

Let $\rho = P_N\eta - V$ and $\theta = P_N u - S$. Proceeding as in the other sections, we obtain the following system

$$\begin{aligned}(\rho_t, \varphi) + (\theta_x, \varphi) + (\eta_x(u - P_N u) + \eta_x\theta + u_x(\eta - P_N\eta) + u_x\rho, \varphi) + \tfrac{1}{6}(\theta_{xxx}, \varphi) = 0,\\ (\theta_t, \psi) + (\rho_x, \psi) + (u_x(u - P_N u) + u_x\theta, \psi) + \tfrac{1}{6}(\rho_{xxx}, \psi) = 0.\end{aligned} \tag{4.99}$$

Taking $\varphi = \rho$ and $\psi = \theta$ in (4.99) and using the Schwarz and Young's inequalities, we obtain

$$\frac{d}{dt}\left(\|\rho\|^2 + \|\theta\|^2\right) \leq c_2\left(\|\rho\|^2 + \|\theta\|^2\right) + c_3, \tag{4.100}$$

where $c_2 = \max\ \{3 + 2\|u_x\|_\infty, \|\eta_x\|_\infty^2 + 2\|u_x\|_\infty\}$ and $c_3 = \|\eta_x\|_\infty^2\|u - P_N u\|^2 + \|u_x\|_\infty^2$ $\|\eta - P_N\eta\|^2 + \|u_x\|_\infty^2\|u - P_N u\|^2$ are constants such that, $c_2 \leq CN^{-2s}$ and c_3 is bounded, as consequences of Proposition 4 and (4.36).

Then, integrating (4.100) over $[0, t]$ and using the Gronwall's lemma, we obtain that

$$\max_{0\leq t\leq T}\left(\|\rho\| + \|\theta\|\right) \leq C(T)N^{-s}. \tag{4.101}$$

The Lemma follows by Proposition 4 and (4.101). □

Theorem 19. *Let $(\eta, u) \in C(0, T; H^s(\Omega) \times H^s(\Omega))$ be the solution of (4.92) corresponding to the initial data $(\eta_0, u_0) \in H^s(\Omega) \times H^s(\Omega)$ for $s \geq 2$. Then, there exists a finite time $T > 0$ and a constant $C = C(T) > 0$ independent of N, such that*

$$\max_{0\leq t\leq T}\left(\|\eta(t) - \eta_N\| + \|u(t) - u_N\|\right) \leq CN^{-s}. \tag{4.102}$$

Proof. As in the previous sections, we define $e_1 = V - \eta_N$ and $e_2 = S - u_N$. Using these definitions and taking into account that

$$\begin{aligned}\eta_x S + u_x V - (\eta_N u_N)_x = &- \eta_x(u - S) + (\eta u)_x - u_x(\eta - V) - (V_x - e_{1_x})(S - e_2)\\ &- (S_x - e_{2_x})(V - e_1),\end{aligned}$$

and

$$u_x S - u_N u_{N_x} = -u_x(u-S) + uu_x - (S_x - e_{2_x})(S - e_2),$$

we can write the following system

$$\begin{aligned}
&(e_{1_t}, \varphi) + (e_{2_x}, \varphi) + (-\eta_x(u-S) + (\eta u)_x - u_x(\eta - V) - (V_x - e_{1_x})(S - e_2), \varphi) \\
&\quad -((S_x - e_{2_x})(V - e_1), \varphi) + \tfrac{1}{6}(e_{2_{xxx}}, \varphi) = 0, \\
&(e_{2_t}, \psi) + (e_{1_x}, \psi) + (-u_x(u-S) + uu_x - (S_x - e_{2_x})(S - e_2), \psi) + \tfrac{1}{6}(e_{1_{xxx}}, \psi) = 0,
\end{aligned} \tag{4.103}$$

supplemented with the initial conditions $e_1(0) = e_2(0) = 0$.

Let $t_h \in (0, T]$ be the maximal temporal instance for which (4.103) has a unique solution such that

$$\|e_{2_x}(t)\|_\infty \leq 1, \quad \text{for} \quad 0 \leq t \leq t_h. \tag{4.104}$$

Then, if we take $\varphi = e_1$ and $\psi = e_2$ in (4.103) and use the Schwarz and Young's inequalities, we obtain that

$$\frac{d}{dt}(\|e_1\|^2 + \|e_2\|^2) \leq c_4(\|e_1\|^2 + \|e_2\|^2) + c_5, \tag{4.105}$$

where c_4 is a bounded constant and $c_5 \leq CN^{-2s}$. These limitation are consequences of Lemma 7, (4.36), (4.51), (4.52) and (4.53).

Once again, integrating (4.105) over $[0, t]$ and using the Gronwall's lemma, we conclude

$$\max_{0 \leq t \leq T} (\|e_1(t)\| + \|e_2(t)\|) \leq C(T) N^{-s}. \tag{4.106}$$

which is valid for all $t \in [0, t_h]$.

To finish this demonstration, we observe that the Sobolev's lemma, the inverse estimates for elements of S_N given by (4.7), along with (4.106) give us that

$$\|e_{2_x}(t)\|_\infty \leq C_s \|e_2(t)\|_1^{1/2} \|e_2(t)\|_2^{1/2} \leq CN^{\frac{3}{2}} \|e_2(t)\| \leq CN^{\frac{3}{2}-s}, \tag{4.107}$$

for all $t \in [0, t_h]$.

However, since we are considering $s \geq 2$, the term $N^{\frac{3}{2}-s}$ goes to 0 as N approaches to infinity independently of t, which implies that (4.107) holds for all t. Therefore, we can take $t_h = T$ in (4.104). The (4.102) follows by Lemma 7 and (4.106). $\square$

4.3.1.6 H has order 2

For this case, we assume that $b = 0$, $d > 0$, $a < 0$ and $c = 0$. Then, the Boussinesq system becomes:

$$\begin{aligned} \eta_t + u_x + (\eta u)_x + a u_{xxx} &= 0, \\ u_t + \eta_x + u u_x - d u_{xxt} &= 0, \end{aligned} \tag{4.108}$$

with initial conditions $\eta(x, 0) = \eta_0(x)$ and $u(x, 0) = u_0(x)$ for $x \in \Omega$ and $t \geq 0$.

In order to obtain error bounds for the semi discrete problem obtained from (4.108) using the Fourier collocation method, we consider weight functions φ, ψ as functions from S_N as well, and require that the following system is satisfied

$$\begin{aligned} (\eta_{N_t}, \varphi) + (u_{N_x}, \varphi) + ((\eta_N u_N)_x, \varphi) + a(u_{N_{xxx}}, \varphi) &= 0, \\ (u_{N_t}, \psi) + (\eta_{N_x}, \psi) + (u_N u_{N_x}, \psi) - d(u_{N_{xxt}}, \psi) &= 0, \end{aligned} \tag{4.109}$$

in Ω and for $t \geq 0$. The system (4.109) is supplemented with the initial conditions $\eta_N(x, 0) = P_N \eta_0(x)$ and $u_N(x, 0) = P_N u_0(x)$, for $x \in Omega$.

Moreover, as it was seen on the Section 3.3.2, the inital condition and the solution in this case satisfy that $(\eta_0, u_0) \in H^s(\Omega) \times H^{s+2}(\Omega)$ and $(\eta, u) \in C(0, T; H^s(\Omega) \times H^{s+2}(\Omega))$ for $s \geq 1$ and $T > 0$.

We proceed similarly to the previous sections. For the first part, given $(\eta, u) \in C(0, T; H^s(\Omega) \times H^{s+2}(\Omega))$ solution of (4.108) corresponding to initial data $(\eta_0, u_0) \in$

$H^s(\Omega) \times H^{s+2}(\Omega)$, we seek for functions $V, S \in S_N$ satisfying for all $\varphi, \psi \in S_N$ the following system

$$\begin{aligned}(V_t, \varphi) + (S_x, \varphi) + ((uV_x + \eta S_x), \varphi) + a(S_{xxx}, \varphi) = 0,\\ (S_t, \psi) + (V_x, \psi) + (uS_x, \psi) - d(S_{xxt}, \psi) = 0,\end{aligned} \tag{4.110}$$

supplemented with initial conditions given by $V(x, 0) = P_N\eta_0(x)$ and $S(x, 0) = P_N u_0(x)$.

We have the following result.

Lemma 8. *Let $(\eta, u) \in C(0, T; H^s(\Omega) \times H^{s+2}(\Omega))$ be the solution of (4.108) corresponding to initial data $(\eta_0, u_0) \in H^s(\Omega) \times H^{s+2}(\Omega)$, for $s \geq 4$. Then, there is a unique solution $(V, S) \in S_N \times S_N$ of (4.110) for all finite time $T \geq 0$. Moreover, given $0 \leq t \leq T$, there is a constant $C = C(T, a, d) > 0$ independent of N, such that*

$$\max_{0\leq t\leq T} \{\|\eta(t) - V\| + \|u(t) - S\|_1\} \leq CN^{1-s}. \tag{4.111}$$

Proof. We consider $\rho = P_N\eta - V$ and $\theta = P_N u - S$, so that (4.28) and (4.5) hold. As in the others cases, we can use the decomposition given by (4.28), (4.30) and (4.31), and we can obtain the following system

$$\begin{aligned}(\rho_t, \varphi) + (\theta_x, \varphi) + (u(\eta - P_N\eta)_x + u\rho_x + \eta(u - P_N u)_x + \eta\theta_x, \varphi) + a(\theta_{xxx}, \varphi) = 0,\\ (\theta_t, \psi) + (\rho_x, \psi) + (u(u - P_N u)_x + u\theta_x, \psi) - d(\theta_{xxt}, \psi) = 0,\end{aligned} \tag{4.112}$$

where $\rho(0) = \theta(0) = 0$.

Let $t_h \in (0, T]$ be the maximal temporal instance for which (4.112) has a unique solution such that

$$\|\rho_{xx}(t)\|_\infty \leq 1, \quad \text{for } \; 0 \leq t \leq t_h. \tag{4.113}$$

Proceeding similar to the previous sections, if we take $\varphi = \rho$ and $\psi = \theta$ in (4.112) and use (4.5), (4.36) along with Gronwall's inequality, we can conclude that

$$\|\rho(t)\| + \|\theta(t)\|_1 \leq C(T, a, d)N^{1-s}, \tag{4.114}$$

for all $t \in [0, t_h]$.

Observe that, by the Sobolev's lemma and the inverse estimates, we get

$$\|\rho_{xx}(t)\|_\infty \leq C\|\rho(t)\|_2^{1/2}\|\rho(t)\|_3^{1/2} \leq CN^{\frac{5}{2}}\|\rho(t)\| \leq CN^{\frac{7}{2}-s}, \tag{4.115}$$

for all $t \in [0, t_h]$, as a consequence of (4.114).

However, since we are considering $s \geq 4$, the term $N^{\frac{7}{2}-s}$ goes to 0 when N approaches to infinity independently of t. Therefore, we can take $t_h = T$ in (4.113), and (4.114) is valid for all $t \in [0, T]$.

The estimate given in (4.114) along with (4.5) implies (4.111).

To conclude this proof, we only have to show that the solution of (4.110) can not blow up, since the existence of an unique local solution is guaranteed by the ODE systems theory. With this aim in mind, we proceed as before taking $\varphi = V$ and $\psi = S$ in (4.110), and observing that since (4.115) holds for all $t \in [0, T]$, we can show that $\|V_{xx}\|_\infty < C$, for $s \geq 4$. This fact ensures that

$$\max_{0\leq t\leq T}(\|V(t)\| + \|S(t)\|_1) \leq C(T, a, d, \|V_0\|, \|S_0\|_1), \tag{4.116}$$

and therefore, the solution of (4.110) can be extend over $[0, T]$. □

For the second part, we prove the following Theorem.

Theorem 20. *Let* $(\eta, u) \in C(0, T; H^s(\Omega) \times H^{s+2}(\Omega))$ *be the solution of (4.55) corresponding to the initial data* $(\eta_0, u_0) \in H^s(\Omega) \times H^{s+2}(\Omega)$*, for* $s \geq 4$*. Then, there exists a finite time* $T > 0$ *and a constant* $C = C(T, a, d) > 0$ *independent of* N*, such that*

$$\max_{0\leq t\leq T}(\|\eta(t) - \eta_N\| + \|u(t) - u_N\|_1) \leq CN^{1-s}. \tag{4.117}$$

Proof. This proof is similar to the proof of Theorem 15. The only difference is that we take $t_h \in (0,T]$ be the maximal temporal instance for which the system

$$\begin{aligned}(e_{1_t}, e_1) + (e_{2_x}, e_1) + (V_x(u-S) + S_x(\eta - V) + (Se_1)_x + (Ve_2)_x - (e_1e_2)_x, e_1)\\ +a(e_{2_{xxx}}, e_1) = 0,\\ (e_{2_t}, e_2) + (e_{1_x}, e_2) + ((u-S)S_x + (Se_2)_x - e_2e_{2_x}, e_2) - d(e_{2_{xxt}}, e_2) = 0,\end{aligned} \tag{4.118}$$

with $e_1(0) = e_2(0) = 0$, has a unique solution such that,

$$\|e_1(t)\|_\infty \le 1, \quad \|e_{1_x}(t)\|_\infty \le 1, \quad \|e_{1_{xx}}(t)\|_\infty \le 1, \tag{4.119}$$

for all $0 \le t \le t_h$.

Using similar ideas to the previous theorems, the estimates given in (4.119) lead us to conclude that

$$\|e_1(t)\| + \|e_2(t)\|_1 \le C(T,a,d)N^{1-s}, \tag{4.120}$$

which holds for all $t \in [0, t_h]$.

To extend this result to $[0,T]$, we proceed as before, using the Sobolev and inverse inequalities. Then, (4.117) follows by (4.120) and the result obtained in Lemma 8. □

4.3.1.7 H has order 1

For this case, we assume that $b = 0$, $d > 0$, $a = c \ge 0$ or $a, c < 0$. Then, the Boussinesq system becomes:

$$\begin{aligned}\eta_t + u_x + (\eta u)_x + au_{xxx} = 0,\\ u_t + \eta_x + uu_x + c\eta_{xxx} - du_{xxt} = 0,\end{aligned} \tag{4.121}$$

with initial conditions $\eta(x,0) = \eta_0(x)$ and $u(x,0) = u_0(x)$ for $x \in \Omega$ and $t \ge 0$.

In order to obtain error bounds for the semi discrete problem obtained from (4.121) using the Fourier collocation method, we consider weight functions φ, ψ as functions from S_N as well, and require that the following system is satisfied

$$\begin{aligned} (\eta_{N_t}, \varphi) + (u_{N_x}, \varphi) + ((\eta_N u_N)_x, \varphi) + a(u_{N_{xxx}}, \varphi) = 0, \\ (u_{N_t}, \psi) + (\eta_{N_x}, \psi) + (u_N u_{N_x}, \psi) + c(\eta_{N_{xxx}}, \psi) - d(u_{N_{xxt}}, \psi) = 0, \end{aligned} \tag{4.122}$$

in Ω and for $t \geq 0$. The system (4.122) is supplemented with the initial conditions $\eta_N(x,0) = P_N \eta_0(x)$ and $u_N(x,0) = P_N u_0(x)$, for $x \in \Omega$.

Moreover, as it was seen on the Section 3.3.2, the inital condition and the solution in this case satisfy that $(\eta_0, u_0) \in H^s(\Omega) \times H^{s+1}(\Omega)$ and $(\eta, u) \in C(0,T; H^s(\Omega) \times H^{s+1}(\Omega))$ for $s \geq 1$ and $T > 0$.

Once again, following the idea of dividing the analysis in two parts, we have the following results.

Lemma 9. *Let $(\eta, u) \in C(0,T; H^s(\Omega) \times H^{s+1}(\Omega))$ be the solution of (4.88) corresponding to initial data $(\eta_0, u_0) \in H^s(\Omega) \times H^{s+1}(\Omega)$ for $s \geq 2$. Then, there is a unique solution $(V, S) \in S_N \times S_N$ of*

$$\begin{aligned} (V_t, \varphi) + (S_x, \varphi) + ((uV_x + \eta S_x), \varphi) + a(S_{xxx}, \varphi) = 0, \\ (S_t, \psi) + (V_x, \psi) + (uS_x, \psi)^+ c(V_{xxx}, \psi) - d(S_{xxt}, \psi) = 0, \end{aligned}$$

for all finite time $T \geq 0$. Moreover, given $0 \leq t \leq T$, there is a constant $C = C(T, a, c, d) > 0$ independent of N, such that

$$\max_{0 \leq t \leq T} \{\|\eta(t) - V\| + \|u(t) - S\|_1\} \leq CN^{1-s}. \tag{4.123}$$

Theorem 21. *Let $(\eta, u) \in C(0,T; H^s(\Omega) \times H^{s+1}(\Omega))$ be the solution of (4.121) corresponding to the initial data $(\eta_0, u_0) \in H^s(\Omega) \times H^{s+1}(\Omega)$, for $s \geq 2$. Then, there exists a finite time $T > 0$ and a constant $C = C(T, a, c, d) > 0$ independent of N, such that*

$$\max_{0 \leq t \leq T} (\|\eta(t) - \eta_N\| + \|u(t) - u_N\|_1) \leq CN^{1-s}. \tag{4.124}$$

The proofs of these results are analogous to the ones presented on Sections 4.3.1.1 and 4.3.1.2. In this case, we have to notice that Remark 6 still holds if $a = c \geq 0$ or $a, c < 0$.

4.3.1.8 H has order 0

For this case, we assume that $b > 0, d = 0$ or $b = 0, d > 0$ and $a < 0, c = 0$ or $a = 0, c < 0$. Suppose without loss of generality that $b > 0, d = 0$ and $a < 0, c = 0$. Then, the Boussinesq system in this case becomes

$$\begin{aligned} \eta_t + u_x + (\eta u)_x + a u_{xxx} - b\eta_{xxt} &= 0, \\ u_t + \eta_x + u u_x &= 0, \end{aligned} \tag{4.125}$$

with initial conditions $\eta(x,0) = \eta_0(x)$ and $u(x,0) = u_0(x)$ for $x \in \Omega$ and $t \geq 0$.

In order to obtain error bounds for the semi discrete problem obtained from (4.125) using the Fourier collocation method, we consider weight functions φ, ψ as functions from S_N as well, and require that the following system is satisfied

$$\begin{aligned} (\eta_{N_t}, \varphi) + (u_{N_x}, \varphi) + ((\eta_N u_N)_x, \varphi) + a(u_{N_{xxx}}, \varphi) - b(\eta_{N_{xxt}}, \varphi) &= 0, \\ (u_{N_t}, \psi) + (\eta_{N_x}, \psi) + (u_N u_{N_x}, \psi) &= 0, \end{aligned} \tag{4.126}$$

in Ω and for $t \geq 0$. The system (4.126) is supplemented with the initial conditions $\eta_N(x,0) = P_N\eta_0(x)$ and $u_N(x,0) = P_N u_0(x)$, for $x \in \Omega$.

Moreover, as it was seen on the Section 3.3.2, the inital condition and the solution in this case satisfy that $(\eta_0, u_0) \in H^s(\Omega) \times H^s(\Omega)$ and $(\eta, u) \in C(0,T; H^s(\Omega) \times H^s(\Omega))$ for $s \geq 2$ and $T > 0$.

Therefore, we have the following results

Lemma 10. *Let $(\eta, u) \in C(0,T; H^s(\Omega) \times H^s(\Omega))$ be the solution of (4.125) corresponding to initial data $(\eta_0, u_0) \in H^s(\Omega) \times H^s(\Omega)$ for $s \geq 4$. Then, there is a unique* *solution* $(V, S) \in S_N \times S_N$ *of the system*

$$\begin{aligned} (V_t, \varphi) + (S_x, \varphi) + ((uV_x + \eta S_x), \varphi) + a(S_{xxx}, \varphi) - b(V_{xxt}, \varphi) = 0, \\ (S_t, \psi) + (V_x, \psi) + (uS_x, \psi) = 0, \end{aligned} \tag{4.127}$$

with initial conditions $V(x, 0) = P_N\eta_0(x)$ *and* $S(x, 0) = P_N u_0(x)$*, for all finite time* $T \geq 0$*. Moreover, given* $0 \leq t \leq T$*, there is a constant* $C = C(T, a, b) > 0$ *independent of* N*, such that*

$$\max_{0 \leq t \leq T} \{\|\eta(t) - V\|_1 + \|u(t) - S\|\} \leq CN^{1-s}. \tag{4.128}$$

Theorem 22. *Let* $(\eta, u) \in C(0, T; H^s(\Omega) \times H^s(\Omega))$ *be the solution of (4.125) corresponding to the initial data* $(\eta_0, u_0) \in H^s(\Omega) \times H^s(\Omega)$ *for* $s \geq 4$*. Then, there exists a finite time* $T > 0$ *and a constant* $C = C(T, a, b) > 0$ *independent of* N*, such that*

$$\max_{0 \leq t \leq T} (\|\eta(t) - \eta_N\|_1 + \|u(t) - u_N\|) \leq CN^{1-s}. \tag{4.129}$$

The proofs of Lemma 10 and Theorem 22 are analogous to the proofs of Lemma 8 and Theorem 20 on Section 4.3.1.6, respectively. Assumptions were made about $\|\theta_{xx}(t)\|_\infty$, $\|e_2(t)\|_\infty$, $\|e_{2_x}(t)\|_\infty$ and $\|e_{2_{xx}}(t)\|_\infty$ rather than the norms of $\|\rho(t)\|$ and $\|e_1(t)\|$.

4.3.1.9 H has order −1

For this case, we assume that $b > 0$, $d = 0$ and $a = c \geq 0$. Then, the Boussinesq system becomes:

$$\begin{aligned} \eta_t + u_x + (\eta u)_x + au_{xxx} - b\eta_{xxt} = 0, \\ u_t + \eta_x + uu_x + c\eta_{xxx} = 0, \end{aligned} \tag{4.130}$$

with initial conditions $\eta(x, 0) = \eta_0(x)$ and $u(x, 0) = u_0(x)$ for $x \in \Omega$ and $t \geq 0$.

In order to obtain error bounds for the semi discrete problem obtained from (4.130) using the Fourier collocation method, we consider weight functions φ, ψ as functions from S_N as well, and require that the following system is satisfied

$$\begin{aligned}(\eta_{N_t},\varphi)+(u_{N_x},\varphi)+((\eta_N u_N)_x,\varphi)+a(u_{N_{xxx}},\varphi)-b(\eta_{N_{xxt}},\psi)=0,\\ (u_{N_t},\psi)+(\eta_{N_x},\psi)+(u_N u_{N_x},\psi)+c(\eta_{N_{xxx}},\psi)=0,\end{aligned} \tag{4.131}$$

in Ω and for $t\geq 0$. The system (4.131) is supplemented with the initial conditions $\eta_N(x,0)=P_N\eta_0(x)$ and $u_N(x,0)=P_N u_0(x)$, for $x\in\Omega$.

Moreover, as it was seen on the Section 3.3.2, the inital condition and the solution in this case satisfy that $(\eta_0,u_0)\in H^{s+1}(\Omega)\times H^s(\Omega)$ and $(\eta,u)\in C(0,T;H^{s+1}(\Omega)\times H^s(\Omega))$ for $s\geq 1$ and $T>0$.

Following the idea of dividing the analysis in two parts, we have the following results.

Lemma 11. *Let $(\eta,u)\in C(0,T;H^{s+1}(\Omega)\times H^s(\Omega))$ be the solution of (4.130) corresponding to initial data $(\eta_0,u_0)\in H^{s+1}(\Omega)\times H^s(\Omega)$ for $s\geq 2$. Then, there is a unique solution $(V,S)\in S_N\times S_N$ of*

$$\begin{aligned}(V_t,\varphi)+(S_x,\varphi)+((uV_x+\eta S_x),\varphi)+a(S_{xxx},\varphi)-b(V_{xxt},\psi)=0,\\ (S_t,\psi)+(V_x,\psi)+(uS_x,\psi)^{+}c(V_{xxx},\psi)=0,\end{aligned}$$

for all finite time $T\geq 0$. Moreover, given $0\leq t\leq T$, there is a constant $C=C(T,a,b,c)>0$ independent of N, such that

$$\max_{0\leq t\leq T}\ \{\|\eta(t)-V\|_1+\|u(t)-S\|\}\leq CN^{1-s}. \tag{4.132}$$

Theorem 23. *Let $(\eta,u)\in C(0,T;H^{s+1}(\Omega)\times H^s(\Omega))$ be the solution of (4.130) corresponding to the initial data $(\eta_0,u_0)\in H^{s+1}(\Omega)\times H^s(\Omega)$ for $s\geq 2$. Then, there exists a finite time $T>0$ and a constant $C=C(T,a,b,c)>0$ independent of N, such that*

$$\max_{0\leq t\leq T}(\|\eta(t)-\eta_N\|_1+\|u(t)-u_N\|)\leq CN^{1-s}. \tag{4.133}$$

The proofs of these results are analogous to the Section 4.3.1.7. In this case, once again, we observe that Remark 6 still holds if $a=c\geq 0$. The roles of V and

S and ρ and θ are reversed, during the demonstration of the Lemma 11, as well as, the roles of e_1 and e_2, during the demonstration of Theorem 23.

4.3.2 Fully Discrete Problem

In this section, considering the time and space variables discretized, we prove optimal order L^2, H^1 error bounds of spectral accuracy in $\Omega = [-L, L]$ and fourth order accuracy in time for the nonlinear family of Boussinesq systems (3.31).

As it was mentioned in the beginning of this chapter, the nonlinear systems are discretized in space by the standard Fourier collocation spectral method and in time by the explicit RK4 method. The algorithm for the RK4 method is detailed in chapter 2.

In general, the semi discrete problem for the Boussinesq system is represented by

$$\begin{aligned} (\eta_{N_t}, \varphi) + (u_{N_x}, \varphi) + ((\eta_N u_N)_x, \varphi) + a(u_{N_{xxx}}, \varphi) - b(\eta_{N_{xxt}}, \varphi) = 0, \\ (u_{N_t}, \psi) + (\eta_{N_x}, \psi) + (u_N u_{N_x}, \psi) + c(\eta_{N_{xxx}}, \psi) - d(u_{N_{xxt}}, \psi) = 0, \end{aligned} \tag{4.134}$$

valid for all $t \in [0, T]$, where the functions φ, ψ are the weight function of the Fourier collocation method and the parameters a, b, c, d are given by (1.13). The form of (4.134) changes depending on the region of parameters considered.

We start proving the following result, which will be necessary to obtain the mentioned estimates.

Lemma 12. *For N sufficiently large, let (η_N, u_N) be the solution of the semi discrete problem given by (4.134) for $t \in [0, T]$. Therefore, for $j = 0, 1, 2, 3, 4$, there exist constants C_j independent of N, such that* *(i) Purely BBM-type: if* $s \geq 2$ *is such that* $(\eta, u) \in C(0,T;H^s(\Omega))^2$ *and* $(\partial_t^k \eta, \partial_t^k u) \in C(0,T;H^{s+1}(\Omega))^2$ *for* $k \geq 1$*, then*

$$\max_{0 \leq t \leq T} \left(\|\partial_t^j \eta_N\|_1 + \|\partial_t^j u_N\|_1 \right) \leq C_j.$$

(ii.1) Weakly Dispersive case I: if $s \geq 3$ *is such that* $(\eta, u) \in C(0,T;H^s(\Omega))^2$ *and* $(\partial_t \eta, \partial_t u) \in C(0,T;H^{s-1}(\Omega))^2$*, then*

$$\max_{0 \leq t \leq T} \left(\|\partial_t^j \eta_N\|_1 + \|\partial_t^j u_N\|_1 \right) \leq C_j.$$

(ii.2) Weakly Dispersive case II: if $s \geq 3$ *is such that* $(\eta, u) \in C(0,T;H^{s+1}(\Omega) \times H^s(\Omega))$ *and* $(\partial_t \eta, \partial_t u) \in C(0,T;H^{s+1}(\Omega) \times H^s(\Omega))$*, then*

$$\max_{0 \leq t \leq T} \left(\|\partial_t^j \eta_N\|_2 + \|\partial_t^j u_N\|_1 \right) \leq C_j.$$

(ii.3) Weakly Dispersive case III: if $s \geq 3$ *is such that* $(\eta, u) \in C(0,T;H^s(\Omega) \times H^{s+1}(\Omega))$ *and* $(\partial_t \eta, \partial_t u) \in C(0,T;H^s(\Omega) \times H^{s+1}(\Omega))$*, then*

$$\max_{0 \leq t \leq T} \left(\|\partial_t^j \eta_N\|_1 + \|\partial_t^j u_N\|_2 \right) \leq C_j.$$

(iii) Purely KdV-type: if $s \geq 5$ *is such that* $(\eta, u) \in C(0,T;H^s(\Omega))^2$ *and* $(\partial_t \eta, \partial_t u) \in C(0,T;H^{s-3}(\Omega))^2$*, then*

$$\max_{0 \leq t \leq T} \left(\|\partial_t^j \eta_N\| + \|\partial_t^j u_N\| \right) \leq C_j.$$

(iv) H of order 2: if $s \geq 4$ *is such that* $(\eta, u) \in C(0,T;H^s(\Omega) \times H^{s+2}(\Omega))$ *and* $(\partial_t \eta, \partial_t u) \in C(0,T;H^{s-1}(\Omega) \times H^{s+1}(\Omega))$*, then*

$$\max_{0 \leq t \leq T} \left(\|\partial_t^j \eta_N\| + \|\partial_t^j u_N\|_1 \right) \leq C_j.$$

(v) H of order 1: if $s \geq 4$ *is such that* $(\eta, u) \in C(0,T;H^s(\Omega) \times H^{s+1}(\Omega))$ *and* $(\partial_t \eta, \partial_t u) \in C(0,T;H^{s-2}(\Omega) \times H^{s-1}(\Omega))$*, then*

$$\max_{0 \leq t \leq T} \left(\|\partial_t^j \eta_N\| + \|\partial_t^j u_N\|_1 \right) \leq C_j.$$

(vi) H of order 0: if $s \geq 3$ is such that $(\eta, u) \in C(0,T; H^s(\Omega))^2$ and $(\partial_t \eta, \partial_t u) \in C(0,T; H^{s-1}(\Omega))^2$, then

$$\max_{0 \leq t \leq T} \left(\|\partial_t^j \eta_N\|_1 + \|\partial_t^j u_N\| \right) \leq C_j.$$

(vii) H of order -1: if $s \geq 3$ is such that $(\eta, u) \in C(0,T; H^{s+1}(\Omega) \times H^s(\Omega))2$ and $(\partial_t \eta, \partial_t u) \in C(0,T; H^{s-1}(\Omega) \times H^{s-2}(\Omega))$, then

$$\max_{0 \leq t \leq T} \left(\|\partial_t^j \eta_N\|_1 + \|\partial_t^j u_N\| \right) \leq C_j.$$

Proof. (i) Observe that, using the definition of ρ, θ, e_1 e e_2 given on section 4.3.1, we can write

$$\eta_N = P_N \eta - \rho - e_1, \qquad u_N = P_N u - \theta - e_2. \tag{4.135}$$

Therefore, using the estimates for the semi discrete problem in this case, the properties of P_N and the inverse estimates for elements of S_N, we get

$$\begin{aligned} \|\eta_N\|_1 &= \|P_N \eta - \rho - e_1 + \eta - \eta\|_1 \leq \|P_N \eta - \eta\|_1 + \|\eta\|_1 + \|\rho\|_1 + \|e_1\|_1 \\ &\leq C\left(N^{1-s}\|\eta\|_s + \|\eta\|_s + N^{1-s}\right), \\ \|u_N\|_1 &= \|P_N u - \theta - e_2 + u - u\|_1 \leq \|P_N u - u\|_1 + \|u\|_1 + \|\theta\|_1 + \|e_2\|_1 \\ &\leq C\left(N^{1-s}\|u\|_s + \|u\|_s + N^{1-s}\right), \end{aligned}$$

which are both bounded independently of N, for N sufficiently large, $t \in [0,T]$ and $s \geq 2$.

Then, we can conclude that

$$\max_{0 \leq t \leq T} \left(\|\eta_N\|_1 + \|u_N\|_1 \right) \leq C_0, \tag{4.136}$$

with C_0 independent of N.

Now, taking $\varphi = \eta_N$ and $\psi = u_N$ in (4.25), we have that

$$\begin{aligned} a_B(\eta_{N_t}, \eta_{N_t}) &= -2(u_{N_x}, \eta_N) - 2((\eta_N u_N)_x, \eta_N), \\ a_D(u_{N_t}, u_{N_t}) &= -2(\eta_{N_x}, u_N) - 2(u_N u_{N_x}, u_N), \end{aligned}$$

which implies, using the fact that both bilinear forms are coercive, that

$$\begin{aligned}\|\eta_{N_t}\|_1^2 + \|u_{N_t}\|_1^2 &\le 2\|u_{N_x}\|\|\eta_N\| + 3\|u_{N_x}\|_\infty\|\eta_N\|^2 + 2\|\eta_{N_x}\|\|u_N\| + 2\|u_{N_x}\|_\infty\|u_N\|^2 \\ &\le \max\{1, 2+\|u_{N_x}\|_\infty, 1+3\|u_{N_x}\|_\infty\}\left(\|\eta_N\|_1^2 + \|u_N\|_1^2\right) \le C_1,\end{aligned}$$

valid for all $t \in [0,T]$, $s \ge 2$ and N sufficiently large. Indeed, using (4.135) and inverse estimates for elements of S_N, we obtain

$$\begin{array}{rcl}\|u_{N_x}\|_\infty & \le & \|(u-P_Nu)_x\|_\infty + \|u_x\|_\infty + \|\theta_x\|_\infty + \|e_{2_x}\|_\infty \\ & \le & C\left(\|u-P_Nu\|_2 + \|u\|_2 + \|\theta\|_2 + \|e_2\|_2\right) \\ & \le & C\left(N^{2-s}\|u\|_s + \|u\|_2 + N\|\theta\|_1 + N\|e_2\|_1\right) \\ & \le & C\left(N^{2-s}\|u\|_s + \|u\|_2 + N^{2-s}\right),\end{array} \tag{4.137}$$

which is bounded independently of N, since we are considering $s \ge 2$. These estimates along with (4.136), lead us to conclude that

$$\max_{0\le t\le T}\left(\|\eta_{N_t}\|_1 + \|u_{N_t}\|_1\right) \le C_1, \tag{4.138}$$

with C_1 independent of N.

Taking the first time derivative of (4.25) and $\varphi = \eta_{N_t}$ and $\psi = u_{N_t}$ we have

$$\begin{array}{c}(\eta_{N_{tt}}, \eta_{N_t}) + (u_{N_{xt}}, \eta_{N_t}) + ((\eta_N u_N)_{xt}, \eta_{N_t}) - b(\eta_{N_{xxtt}}, \eta_{N_t}) = 0, \\ (u_{N_{tt}}, u_{N_t}) + (\eta_{N_{xt}}, u_{N_t}) + ((u_N u_{N_x})_t, u_{N_t}) - d(u_{N_{xxtt}}, u_{N_t}) = 0.\end{array} \tag{4.139}$$

Proceeding similar to the last estimate, we conclude that

$$\|\eta_{N_{tt}}\|_1^2 + \|u_{N_{tt}}\|_1^2 \le m\left(\|\eta_{N_t}\|_1^2 + \|u_{N_t}\|_1^2\right),$$

where $m = \max\{3+3\|u_{N_x}\|_\infty, 1, 1+\|\eta_{N_x}\|_\infty, 1+3\|u_{N_x}\|_\infty + \|\eta_N\|_\infty\}$.

Using (4.135) and similar arguments used in (4.137), we can show that $\|\eta_N\|_\infty$, $\|u_N\|_\infty$ and $\|\eta_{N_x}\|_\infty$ are all bounded independently of N, since $s \ge 2$. Then, we conclude that m is bounded and it holds that

$$\max_{0\le t\le T}\left(\|\eta_{N_{tt}}\|_1 + \|u_{N_{tt}}\|_1\right) \le C_2, \tag{4.140}$$

with C_2 independent of N.

Taking the second time derivative of (4.25) and $\varphi = \eta_{N_{tt}}$, $\psi = u_{N_{tt}}$, we have

$$\begin{aligned} (\eta_{N_{ttt}}, \eta_{N_{tt}}) + (u_{N_{xtt}}, \eta_{N_{tt}}) + ((\eta_N u_N)_{xtt}, \eta_{N_{tt}}) - b(\eta_{N_{xxttt}}, \eta_{N_{tt}}) = 0, \\ (u_{N_{ttt}}, u_{N_{tt}}) + (\eta_{N_{xtt}}, u_{N_{tt}}) + ((u_N u_{N_x})_{tt}, u_{N_{tt}}) - d(u_{N_{xxttt}}, u_{N_{tt}}) = 0. \end{aligned} \tag{4.141}$$

Once again, similarly to the last estimate, we obtain that

$$\|\eta_{N_{ttt}}\|_1^2 + \|u_{N_{ttt}}\|_1^2 \leq c_1 \left(\|\eta_{N_{tt}}\|_1^2 + \|u_{N_{tt}}\|_1^2\right) + c_2 \left(\|\eta_{N_t}\|_1^2 + \|u_{N_t}\|_1^2\right),$$

with $c_1 = \max\{5 + 2\|u_{N_x}\|_\infty + \|u_N\|_\infty, 1, 2 + 3\|u_{N_x}\|_\infty + \|\eta_{N_x}\|_\infty, 1 + \|\eta_N\|_\infty\}$ and $c_2 = \max\{4 + \|u_{N_{xt}}\|_\infty, 4\|\eta_{N_{xt}}\|_\infty + 4\|u_{N_{xt}}\|_\infty\}$.

Using previous estimates, we conclude that c_1 is a bounded constant independently of N, since $s \geq 2$. For c_2, we observe that, using the same idea as before,

$$\begin{aligned} \|\eta_{N_{xt}}\|_\infty &\leq \|(\eta_t - P_N\eta_t)_x\|_\infty + \|\eta_{tx}\|_\infty + \|\rho_{tx}\|_\infty + \|e_{1_{tx}}\|_\infty \\ &\leq C\left(\|\eta_t - P_N\eta_t\|_2 + \|\eta_t\|_2 + \|\rho_t\|_2 + \|e_{1_t}\|_2\right) \\ &\leq C\left(N^{2-(s+1)}\|\eta_t\|_{s+1} + \|\eta_t\|_2 + N\|\rho_t\|_1 + N\|e_{1_t}\|_1\right) \\ &\leq C\left(N^{1-s}\|\eta_t\|_{s+1} + \|\eta_t\|_2 + N^{2-s}\right), \end{aligned}$$

which is bounded by a constant independent of N since $s \geq 2$ and $(\eta_t, u_t) \in C(0,T;H^{s+1}(\Omega))^2$. We can use a similar argument to bound $\|u_{N_{xt}}\|_\infty$, and conclude that c_2 is, in fact, bounded independently of N.

Using (4.138) and (4.140) along the limitations of c_1 and c_2, we conclude that

$$\max_{0\leq t\leq T}\left(\|\eta_{N_{ttt}}\|_1 + \|u_{N_{ttt}}\|_1\right) \leq C_3, \tag{4.142}$$

with C_3 independent of N.

To finish the proof in this case, we have to estimate the fourth time derivative of (η_N, u_N). We take the third time derivative of (4.25) and take $\varphi = \eta_{N_{ttt}}$ and

$\psi = u_{N_{ttt}}$. Then, we obtain

$$\begin{aligned} a_B(\eta_{N_{tttt}}, \eta_{N_{ttt}}) + (u_{N_{xttt}}, \eta_{N_{ttt}}) + ((\eta_N u_N)_{xttt}, \eta_{N_{ttt}}) + a(u_{N_{xxxttt}}, \eta_{N_{ttt}}) = 0, \\ a_D(u_{N_{tttt}}, u_{N_{ttt}}) + (\eta_{N_{xttt}}, u_{N_{ttt}}) + ((u_N u_{N_x})_{ttt}, u_{N_{ttt}}) + c(\eta_{N_{xxxttt}}, u_{N_{ttt}}) = 0. \end{aligned} \tag{4.143}$$

Through the Sobolev's lemma and the limitations of first three time derivatives, we can show that $\|\eta_{N_t}\|_\infty$ and $\|u_{N_t}\|_\infty$ are both bounded independently of N for $s \geq 2$. Then, we conclude that

$$\max_{0\leq t\leq T} (\|\eta_{N_{tttt}}\|_1 + \|u_{N_{tttt}}\|_1) \leq C_4, \tag{4.144}$$

with C_4 independent of N.

Therefore, from (4.136), (4.138), (4.140), (4.142) and (4.144) we can conclude that for $s \geq 2$ and $j = 0, 1, 2, 3, 4$, it holds

$$\max_{0\leq t\leq T} \left(\|\partial_t^j \eta_N\|_1 + \|\partial_t^j u_N\|_1\right) \leq C_j.$$

(ii.1) Since the case *(i)* is a particular case of this one with $a = c = 0$, and if we multiply the first equation by $|c|$ and the second by $|a|$ we can use Remark 6 to cancel these extra terms, all the estimates for the first three times derivatives of (η_N, u_N) given in the previous proof hold in this case. It is also worth to observe that in this case,

$$\begin{aligned} \|\eta_{N_{xt}}\|_\infty &\leq \|(\eta_t - P_N\eta_t)_x\|_\infty + \|\eta_{tx}\|_\infty + \|\rho_{tx}\|_\infty + \|e_{1_{tx}}\|_\infty \\ &\leq C\left(\|\eta_t - P_N\eta_t\|_2 + \|\eta_t\|_2 + \|\rho_t\|_2 + \|e_{1_t}\|_2\right) \\ &\leq C\left(N^{2-(s-1)}\|\eta_t\|_{s-1} + \|\eta_t\|_2 + N\|\rho_t\|_1 + N\|e_{1_t}\|_1\right) \\ &\leq C\left(N^{3-s}\|\eta_t\|_{s-1} + \|\eta_t\|_2 + N^{2-s} + N^{2-s}\right), \end{aligned} \tag{4.145}$$

is bounded by a constant independent of N if $s \geq 3$, since $(\eta_t, u_t) \in C(0, T; H^{s-1}(\Omega))^2$. This implies that the estimate for the fourth time derivative holds for $s \geq 3$.

Therefore, we can conclude that for $s \geq 3$ and $j = 0, 1, 2, 3, 4$, it holds

$$\max_{0\leq t\leq T} \left(\|\partial_t^j \eta_N\|_1 + \|\partial_t^j u_N\|_1\right) \leq C_j$$

(ii.2) Using the decomposition given by (4.135) and using the results for the semi discrete problem for this case, the properties of P_N and the inverse estimates of elements of S_N, we get

$$\begin{aligned}
\|\eta_N\|_2 &= \|P_N\eta - \rho - e_1 + \eta - \eta\|_2 \leq \|P_N\eta - \eta\|_2 + \|\eta\|_2 + \|\rho\|_2 + \|e_1\|_2 \\
&\leq C\left(N^{1-s}\|\eta\|_{s+1} + \|\eta\|_{s+1} + N^{2-s}\right), \\
\|u_N\|_1 &= \|P_N u - \theta - e_2 + u - u\|_1 \leq \|P_N u - u\|_1 + \|u\|_1 + \|\theta\|_1 + \|e_2\|_1 \\
&\leq C\left(N^{1-s}\|u\|_s + \|u\|_s + N^{1-s}\right),
\end{aligned}$$

which are both bounded by a constant independently of N for all $t \in [0,T]$, $s \geq 2$ and N sufficiently large. Then,

$$\max_{0\leq t\leq T}\left(\|\eta_N\|_2 + \|u_N\|_1\right) \leq C_0, \tag{4.146}$$

with C_0 independent of N.

Proceeding similar to the proof of Lemma 5, we write the first equation of (4.70) as

$$(\eta_{N_t} - b\eta_{N_{xtt}}, \varphi) = -(u_{N_x} + (\eta_N u_N)_x, \varphi),$$

with $\eta_{N_t} = P_N\zeta$, such that ζ is solution of $\zeta - b\zeta'' = f(x)$, with $f(x) = -u_{N_x} - (\eta_N u_N)_x$; by (4.73) and (4.74) we obtain

$$\|\eta_{N_t}\|_2 \leq C\left(\|u_{N_x}\| + \|\eta_{N_x}\|_\infty\|u_N\| + \|u_{N_x}\|_\infty\|\eta_N\|\right).$$

On the other hand, taking $\psi = u_N$ in the second equation of (4.70) and using the coercivity of the bilinear form a_D, we obtain that

$$\|u_{N_t}\|_1^2 \leq \|\eta_{N_x}\|^2 + \|u_N\|^2 + 2\|u_{N_x}\|_\infty\|u_N\|^2 + c^2\|\eta_{N_{xx}}\|^2 + \|u_{N_x}\|^2.$$

Adding both inequalities, we conclude that

$$\|\eta_{N_t}\|_2 + \|u_{N_t}\|_1 \leq k\left(\|\eta_N\|_2 + \|u_N\|_1\right),$$

where $k = \max\left\{C\|u_{N_x}\|_\infty, 1, c, C\|\eta_{N_x}\|_\infty + 1 + (2\|u_{N_x}\|_\infty)^{\frac{1}{2}}, 1+C\right\}$ is a bounded constant independently of N for $s \geq 3$, as a consequence of similar arguments used in (4.137). Therefore,

$$\max_{0\leq t\leq T}\left(\|\eta_{N_t}\|_2 + \|u_{N_t}\|_1\right) \leq C_1, \tag{4.147}$$

with C_1 independent of N.

To estimate the second time derivative, we take the first time derivative of (4.25) and proceed as before. We get that

$$\|\eta_{N_{tt}}\|_2 + \|u_{N_{tt}}\|_1 \leq k_1\left(\|\eta_{N_t}\|_2 + \|u_{N_t}\|_1\right),$$

where $k_1 = \max\left\{C\|u_{N_x}\|_\infty, C\|u_N\|_\infty + 1, 1, C\|\eta_{N_x}\|_\infty + 1 + (2\|u_{N_x}\|_\infty)^{\frac{1}{2}}, 1+ \|\eta_N\|_\infty + \|u_N\|_\infty + |c|\right\}$ is also a bounded constant independently of N for $s \geq 3$, by similar arguments used in (4.137). Therefore,

$$\max_{0\leq t\leq T}\left(\|\eta_{N_{tt}}\|_2 + \|u_{N_{tt}}\|_1\right) \leq C_2, \tag{4.148}$$

with C_2 independent of N.

To estimate the third and the fourth time derivatives of (η_N, u_N), we have to use along the estimates for η_N, u_N, η_{N_x} and u_{N_x} in $L^\infty(\Omega)$, the estimates for $\|\eta_{N_t}\|_\infty$ and $\|u_{N_t}\|_\infty$, as it was done previously. In this case, we have similar bounds for these norms if $s \geq 3$ as in (4.145), which imply that

$$\max_{0\leq t\leq T}\left(\|\eta_{N_{ttt}}\|_2 + \|u_{N_{ttt}}\|_1\right) \leq C_3, \tag{4.149}$$

with C_3 independent of N, and

$$\max_{0\leq t\leq T}\left(\|\eta_{N_{tttt}}\|_2 + \|u_{N_{tttt}}\|_1\right) \leq C_4, \tag{4.150}$$

with C_4 independent of N.

Therefore, we can conclude that for $s \geq 3$ and $j = 0, 1, 2, 3, 4$, it holds

$$\max_{0\leq t\leq T}\left(\|\partial_t^j \eta_N\|_2 + \|\partial_t^j u_N\|_1\right) \leq C_j.$$

For the rest of the cases, the demonstration follows similarly to these last ones. It is always necessary to estimate $\|\eta_N\|_\infty, \|\eta_{N_x}\|_\infty, \|\eta_{N_{xt}}\|_\infty, \|u_N\|_\infty, \|u_{N_x}\|_\infty,$ $\|u_{N_{xt}}\|_\infty$ using the appropriate H^s-norms for each case, in order to determine a bound for the first four times time derivatives of η and u. This ends the proof. $\square$

Now we are ready to perform the convergence analysis for the fully discretization of the nonlinear Boussinesq systems (3.31). Our analysis is concentrated in the cases having the stability condition for the solution of the linear problem, given by $\Delta t \leq C\Delta x$. These cases, along the ones with stability condition of type $\Delta t \leq C$, have the corresponding fully discretizations not stiff, which implies that we can achieve a high accuracy at a low computational cost. These specific cases are the ones with numerical simulations in the following chapter.

We recall that, to satisfy the linear stability condition $\Delta t \leq C\Delta x$ (or equivalently $\Delta t \leq CN^{-1}$), the paramenters a, b, c, d must to belong in one of the following regions:

(i) Weakly dispersive Boussinesq systems, case I: $a < 0, b > 0, c < 0, d > 0$ or $a = c > 0, b > 0, d > 0$;

(ii) $\mathcal{H}$ of order 0: $a < 0, b > 0, c = 0, d = 0$ or $a = 0, b = 0, c < 0, d > 0$;

(iii) $\mathcal{H}$ of order 1: $a = 0, b = 0, c = 0, d > 0$;

(iv) $\mathcal{H}$ of order 2: $a < 0, b = 0, c = 0, d > 0$.

We start analyzing the case (i). Considering the bilinear forms given by (4.23) and the functions $\hat{f}, \hat{g} : L^2 \to S_N$ given, respectively, by $a_B(\hat{f}(v), \chi) = (v, \chi^{'})$ and $a_D(\hat{g}(w), \chi) = (w, \chi^{'})$ for all χ in S_N, we obtain from (4.56) that

$$
\begin{aligned}
\eta_{N_t} &= f(\eta_N, u_N),\\
u_{N_t} &= g(\eta_N, u_N),
\end{aligned}
\tag{4.151}
$$

valid for all $t \in [0,T]$, where $f : H^1 \times H^2 \to S_N$ and $g : H^2 \times H^1 \to S_N$ are given by $f(v,w) = \hat{f}(w) + \hat{f}(vw) + a\hat{f}(w'')$ and $g(v,w) = \hat{g}(v) + \frac{1}{2}\hat{g}(w^2) + c\hat{g}(v'')$. The $'$ and $''$ denote, respectively, the first and the second derivative with respect to spatial variable in this specif functions.

In the following, we introduce the RK4 method that we use to solve the system (4.151). The algorithm for this method is given by

1. Set $\eta_N^0 = P_N\eta_0$ and $u_N^0 = P_N u_0$;
2. For $n = 0, 1, ..., M-1$ do

   2.1 Set $\eta_N^{n,1} = \eta_N^n$ and $u_N^{n,1} = u_N^n$;

   2.2 For $i = 2, 3, 4$ do

   2.2.1 $\eta_N^{n,i} = \eta_N^n + \Delta t\alpha_i f(\eta_N^{n,i-1}, u_N^{n,i-1})$; $u_N^{n,i} = u^n + \Delta t\alpha_i g(\eta_N^{n,i-1}, u_N^{n,i-1})$;

   2.3 $\eta_N^{n+1} = \eta_N^n + \Delta t \sum_{j=1}^{4} \beta_j f(\eta_N^{n,j}, u_N^{n,j})$; $u_N^{n+1} = u_N^n + \Delta t \sum_{j=1}^{4} \beta_j g(\eta_N^{n,j}, u_N^{n,j})$,

with $\alpha_2 = \alpha_3 = \frac{1}{2}$, $\alpha_4 = 1$, $\beta_1 = \beta_4 = \frac{1}{6}$ and $\beta_2 = \beta_3 = \frac{1}{3}$.

Then, we have the following result.

Theorem 24. *Let $(\eta, u) \in C(0,T; H^s(\Omega) \times H^s(\Omega))$ be solution of (4.55) corresponding to the initial data $(\eta_0, u_0) \in H^s(\Omega) \times H^s(\Omega)$, for $s \geq 2$ and some $0 < T < \infty$. Suppose that there exists a constant $\hat{M}$, such that $\max_{t\in[0,T]} \left(\|\partial_t^i \eta(t)\|_1 + \|\partial_t^i u(t)\|_1\right) \leq \hat{M}$ for $i = 0, 1, ..., 5$. Let (H^n, U^n), $0 \leq n \leq M$, be the solution of the system (4.151) obtained by the RK4 method. Then, for N sufficiently large and Δt sufficiently small such that $\Delta t \leq CN^{-1}$, there exists a constant also denoted by C, independent of N and Δt, such that*

$$\max_{0\leq n\leq M} \left(\|\eta(t^n) - H^n\|_1 + \|u(t_n) - U^n\|_1\right) \leq C\left(\Delta t^4 + N^{1-s}\right). \tag{4.152}$$

Proof. For the first part of this proof, we consider the local temporal errors of the RK4 method, which are given by

$$\begin{aligned} \delta_1^n &:= \eta_N^{n+1} - \eta_N^n - \Delta t \sum_{j=1}^{4} \beta_j f(\eta_N^{n,j}, u_N^{n,j}), \\ \delta_2^n &:= u_N^{n+1} - u_N^n - \Delta t \sum_{j=1}^{4} \beta_j g(\eta_N^{n,j}, u_N^{n,j}). \end{aligned} \tag{4.153}$$

In the following, we demonstrate that under our hypotheses,

$$\max_{0 \le n \le M-1} (\|\delta_1^n\|_1 + \|\delta_2^n\|_1) \le C\Delta t^5. \tag{4.154}$$

In fact, this can be done by explicitly computing the intermediate stages of RK4 method $(\eta_N^{n,j}, u_N^{n,j})$ and the values of $f(\eta_N^{n,j}, u_N^{n,j})$ and $g(\eta_N^{n,j}, u_N^{n,j})$ in terms of the temporal derivatives of $\eta_N(t)$ and $u_N(t)$ evaluated at $t = t_n$. Using the definitions of f and g we obtain that,

1. For $i = 1 \Rightarrow$ $\begin{aligned} &\eta_N^{n,1} = \eta_N^n = \eta_N(t_n), \quad u_N^{n,1} = u_N^n = u_N(t_n); \\ &f(\eta_N^{n,1}, u_N^{n,1}) = \eta_{N_t}^n, \quad g(\eta_N^{n,1}, u_N^{n,1}) = u_{N_t}^n; \end{aligned}$

2. For $i = 2 \Rightarrow$ $\begin{aligned} &\eta_N^{n,2} = \eta_N^n + \tfrac{\Delta t}{2}\eta_{N_t}^n, \quad u_N^{n,2} = u_N^n + \tfrac{\Delta t}{2} u_{N_t}^n; \\ &f(\eta_N^{n,2}, u_N^{n,2}) = \eta_{N_t}^n + \tfrac{\Delta t}{2}\eta_{N_{tt}}^n + \tfrac{\Delta t^2}{4}\alpha^n, \quad \alpha^n = \hat{f}(\eta_{N_t} u_{N_t}); \\ &g(\eta_N^{n,2}, u_N^{n,2}) = u_{N_t}^n + \tfrac{\Delta t}{2} u_{N_{tt}}^n + \tfrac{\Delta t^2}{2}\beta^n, \quad \beta^n = \hat{g}(u_{N_t} u_{N_t}); \end{aligned}$

3. For $i = 3 \Rightarrow$ $\begin{aligned} &\eta_N^{n,3} = \eta_N^n + \tfrac{\Delta t}{2}\eta_{N_t}^n + \tfrac{\Delta t^2}{4}\eta_{N_{tt}}^n + \tfrac{\Delta t^3}{8}\alpha^n; \\ &u_N^{n,3} = u_N^n + \tfrac{\Delta t}{2} u_{N_t}^n + \tfrac{\Delta t^2}{4} u_{N_{tt}}^n + \tfrac{\Delta t^3}{16}\beta^n; \\ &f(\eta_N^{n,3}, u_N^{n,3}) = \eta_{N_t}^n + \tfrac{\Delta t}{2}\eta_{N_{tt}}^n + \tfrac{\Delta t^2}{4}\eta_{N_{ttt}}^n - \tfrac{\Delta t^2}{4}\alpha^n + \tfrac{\Delta t^3}{8}\Big\{\hat{f}\,(\alpha^n u_N^n) \\ &\qquad + \tfrac{1}{2} f(\eta_N^n, \beta^n) + \alpha_t^n)\Big\} + \tfrac{\Delta t^4}{16}\gamma_1^n; \\ &g(\eta_N^{n,3}, u_N^{n,3}) = u_{N_t}^n + \tfrac{\Delta t}{2} u_{N_{tt}}^n + \tfrac{\Delta t^2}{4} u_{N_{ttt}}^n - \tfrac{\Delta t^2}{8}\beta^n + \tfrac{\Delta t^3}{8}\{\hat{g}\,(\alpha^n) + \\ &\qquad c\hat{g}(\alpha_{xx}^n) + \tfrac{1}{2}\beta_t^n + \tfrac{1}{2}\hat{g}(\beta^n u_N^n))\Big\} + \tfrac{\Delta t^4}{32}\gamma_2^n, \end{aligned}$

   where

$$\gamma_1^n = \hat{f}\left(\alpha^n\left[u_{N_t}^n + \tfrac{\Delta t}{2}u_{N_{tt}}^n + \tfrac{\Delta t^2}{8}\beta^n\right]\right) + \tfrac{1}{2}\hat{f}\left(\beta^n\left[\eta_{N_t}^n + \tfrac{\Delta t}{2}\eta_{N_{tt}}^n\right]\right) + \hat{f}(\eta_{N_{tt}}^n u_{N_{tt}}^n),$$
$$\gamma_2^n = \hat{g}(u_{N_{tt}}^n u_{N_{tt}}^n) + \hat{g}\left(\beta^n\left[u_{N_t}^n + \tfrac{\Delta t}{2}u_{N_{tt}}^n + \tfrac{\Delta t^2}{16}\beta^n\right]\right);$$

4. For $i = 4 \Rightarrow$

$$\begin{aligned}
&\eta_N^{n,4} = \eta_N^n + \Delta t\eta_{N_t}^n + \tfrac{\Delta t^2}{2}\eta_{N_{tt}}^n + \tfrac{\Delta t^3}{4}\eta_{N_{ttt}}^n - \tfrac{\Delta t^3}{4}\alpha^n + \tfrac{\Delta t^4}{8}\hat{\gamma}_1^n,\\
&\quad \text{with } \hat{\gamma}_1^n = \hat{f}(\alpha^n u_N^n) + \tfrac{1}{2}f(\eta_N^n, \beta^n) + \alpha_t^n + \tfrac{\Delta t}{2}\gamma_1^n;\\
&u_N^{n,4} = u_N^n + \Delta t u_{N_t}^n + \tfrac{\Delta t^2}{2}u_{N_{tt}}^n + \tfrac{\Delta t^3}{4}u_{N_{ttt}}^n - \tfrac{\Delta t^3}{8}\beta^n + \tfrac{\Delta t^4}{8}\hat{\gamma}_2^n,\\
&\quad \text{with } \hat{\gamma}_2^n = \hat{g}(\alpha^n) + c\hat{g}(\alpha_{xx}^n) + \tfrac{1}{2}\beta_t^n + \tfrac{1}{2}\hat{g}(\beta^n u_N^n) + \tfrac{\Delta t}{2}\gamma_2^n;\\
&f(\eta_N^{n,4}, u_N^{n,4}) = \eta_{N_t}^n + \Delta t\eta_{N_{tt}}^n + \tfrac{\Delta t^2}{2}\eta_{N_{ttt}}^n + \tfrac{\Delta t^3}{4}\eta_{N_{tttt}}^n - \tfrac{\Delta t^3}{4}\left\{\alpha_t^n + \right.\\
&\quad \left.\tfrac{1}{2}f(\eta_N^n, \beta^n) + \hat{f}\left(\alpha^n u_N^n\right)\right\} + \tfrac{\Delta t^4}{16}\gamma_3^n;\\
&g(\eta_N^{n,4}, u_N^{n,4}) = u_{N_t}^n + \Delta t u_{N_{tt}}^n + \tfrac{\Delta t^2}{2}u_{N_{ttt}}^n + \tfrac{\Delta t^3}{4}u_{N_{tttt}}^n - \tfrac{\Delta t^3}{4}\left\{\tfrac{1}{2}\beta_t^n\right.\\
&\quad \left. + \hat{g}\left(\alpha^n\right) + c\hat{g}(\alpha_{xx}^n) + \tfrac{1}{2}\hat{g}(u_N^n\beta^n)\right\} + \tfrac{\Delta t^4}{32}\gamma_2^n,
\end{aligned}$$

where

$$\begin{aligned}
\gamma_3^n &= \hat{f}(\hat{\gamma}_2^n) + a\hat{f}(\hat{\gamma}_{2_{xx}}^n) + \hat{f}\left(\hat{\gamma}_1^n\left[u_N^{n,4} - \frac{\Delta t^4}{8}\hat{\gamma}_2^n\right]\right) + \hat{f}\left(\eta_{N_{tt}}^n\left[2u_{N_{tt}}^n + \Delta t u_{N_{ttt}}^n\right.\right.\\
&\left.\left. - \frac{\Delta t}{2}\beta^n\right]\right) + \hat{f}\left(\eta_N^{n,4}\hat{\gamma}_2^n\right) + \hat{f}\left(\eta_{N_t}^n\left[2u_{N_{ttt}}^n - \beta^n\right]\right) + \hat{f}\left(\left[\eta_{N_{ttt}}^n - \alpha^n\right]\left[2u_{N_t}^n\right.\right.\\
&\left.\left. + \Delta t u_{N_{tt}}^n + \frac{\Delta t^2}{2}u_{N_{ttt}}^n - \frac{\Delta t^2}{4}\beta^n\right]\right),\\
\gamma_4^n &= \hat{g}(\hat{\gamma}_1^n) + c\hat{g}(\hat{\gamma}_{1_{xx}}^n) + \hat{g}(u_{N_{tt}}^n u_{N_{tt}}^n) + +\hat{g}\left(\hat{\gamma}_2^n\left[u_N^{n,4} - \frac{\Delta t^4}{16}\hat{\gamma}_2^n\right]\right)\\
&+ \hat{g}\left(\left[u_{N_{ttt}}^n - \frac{1}{2}\beta^n\right]\left[2u_{N_t}^n + \Delta t u_{N_{tt}}^n + \frac{\Delta t^2}{4}u_{N_{ttt}}^n - \frac{\Delta t^2}{8}\beta^n\right]\right).
\end{aligned}$$

Using these formulas in (4.153), we get that

$$\begin{aligned}
\delta_1^n &= \eta_N^{n+1} - \sum_{j=0}^{4}\frac{\Delta t^j}{j!}\partial_t^j\eta_N + \Delta t^5\Gamma_1^n,\\
\delta_2^n &= u_N^{n+1} - \sum_{j=0}^{4}\frac{\Delta t^j}{j!}\partial_t^j u_N + \Delta t^5\Gamma_2^n,
\end{aligned}\tag{4.155}$$

where $\Gamma_1^n = \frac{-1}{48}\left(\gamma_1^n + \gamma_3^n\right)$ and $\Gamma_2^n = \frac{-1}{48}\left(\frac{1}{2}\gamma_2^n + \gamma_4^n\right)$.

Hence, to obtain (4.154), we have to show that $\|\Gamma_1^n\|_1$ and $\|\Gamma_2^n\|_1$ are bounded by constants. With this aim in mind, we resort to the item $(ii.1)$ of Lemma 12 and use the following auxiliary result.

Lemma 13. *There exists a constant C independent of N such that*

(i) $\|\hat{f}(v)\|_1 \leq C\|v\|, \quad \|\hat{g}(v)\|_1 \leq C\|v\|, \ v \in L^2_{per}$;

(ii) $\|\hat{f}(v)\|_2 \leq C\|v\|_1, \quad \|\hat{g}(v)\|_2 \leq C\|v\|_1, \ v \in H^1_{per}$;

(iii) $\|\hat{f}(v)\|_3 \leq C\|v\|_1, \quad \|\hat{g}(v)\|_3 \leq C\|v\|_1, \ v \in H^1_{per}$.

Proof of Lemma 13. To demonstrate (i) we use the definition of $\hat{f}$; taking $\chi = \hat{f}(v) \in S_N$ with $v \in L^2$, we get

$$C\|\hat{f}(v)\|_1^2 \leq a_B(\hat{f}(v), \hat{f}(v)) = (v, \hat{f}_x(v)) \leq \|v\|\|\hat{f}(v)\|_1,$$

which implies that $\|\hat{f}(v)\|_1 \leq C\|v\|$ for all $v \in L^2$. The proof is analogous for $\hat{g}$.

To demonstrate (ii), we consider the BVP $w - bw^{''} = -v^{'}$ for $v \in H^1_{per}$, such that $w^{'}(-L) = w^{'}(L)$. Hence, multiplying this BVP by χ and integrating in Ω, we obtain that

$$\begin{aligned} \left(w - bw^{''}, \chi\right) &= -\left(v^{'}, \chi\right), \\ (w, \chi) + b\left(w^{'}, \chi^{'}\right) &= \left(v, \chi^{'}\right), \\ a_B(w, \chi) &= (v, \chi^{'}). \end{aligned} \tag{4.156}$$

Observe that, using (4.22), we can rewrite (4.156) as $a_B(P_N w, \chi) = (v, \chi^{'})$, and use the definition of $\hat{f}$ to conclude that $P_N w = \hat{f}(v)$. Therefore, using (4.5), we have that

$$\|\hat{f}(v)\|_2 = \|P_N w\|_2 = \|P_N w - w + w\|_2 \leq C\|w\|_2. \tag{4.157}$$

On the other hand, note that

$$\|v\|_1^2 \geq \int_\Omega (v^{'})^2 dx = \int_\Omega (w - bw^{''})^2 d \geq \min\left\{1, 2b, b^2\right\} \|w\|_2^2. \tag{4.158}$$

Then, the equations (4.157) and (4.158) imply that $\|\hat{f}(v)\|_2 \leq C\|v\|_1$ for all $v \in H^1_{per}$. This part of the proof is analogous to $\hat{g}$.

The proof for (iii) is similarly to was done in (ii). Consider the BVP $w^{'} - bw^{'''} = -v^{'}$ for $v \in H^1_{per}$, and $w \in H^4$ such that $w^{''}(-L) = w^{''}(L)$. Hence, multiplying this BVP by $\chi \in S_N$ and integrating in Ω, we obtain that

$$\begin{array}{rcl} \left(w^{'} - bw^{'''}, \chi\right) & = & -\left(v^{'}, \chi\right), \\ \left(w^{'}, \chi\right) + b\left(w^{''}, \chi^{'}\right) & = & \left(v, \chi^{'}\right), \\ a_B(w^{'}, \chi) & = & (v, \chi^{'}). \end{array} \tag{4.159}$$

Using the definition of $\hat{f}$, we obtain that $\hat{f}(v) = w^{'}$. On the other hand, we observe that

$$\begin{array}{rcl} \|v\|_1^2 & \geq & \int_\Omega (v^{'})^2 dx = \int_\Omega (w^{'} - bw^{'''})^2 dx \\ & \geq & \|w^{'}\|^2 + 2b\|w^{''}\|^2 + (b^2 - \varepsilon C_p)\|w^{'''}\|^2 + \varepsilon\|w^{''''}\|^2 \\ & \geq & \min\left\{1, 2b, b^2 - \varepsilon C_p, \varepsilon\right\} \|w^{'}\|_3^2, \end{array} \tag{4.160}$$

which is valid for constants $\varepsilon > 0$ and C_p such that $b^2 - \varepsilon C_p > 0$; the constant C_p arises from the Poincaré inequality applied in the term $\|w^{''''}\|$.

Therefore, we conclude that $\|\hat{f}(v)\|_3 = \|w^{'}\|_3 \leq C\|v\|_1$, with $C = C(b, \varepsilon, C_p)$, which finishes the proof of the lemma. $\square$

Proof of Theorem 24 continuated: To demonstrate that $\|\Gamma_1^n\|_1$ and $\|\Gamma_2^n\|_1$ are bounded by constants, observe that using the Lemmas 12 and 13, we can show that

$$\|\alpha^n\|_1 + \|\beta^n\|_1 \leq \|\hat{f}(\eta^n_{N_t}\eta^n_{N_t})\|_1 + \|\hat{g}(u^n_{N_t}u^n_{N_t})\|_1 \leq C\left(\|\eta^n_{N_t}\|^2 + \|u^n_{N_t}\|^2\right),$$

which is bounded by a constant, as a consequence of $(ii.1)$ of Lemma 12. Moreover,

$$\begin{aligned}\|\gamma_1^n\|_1 + \|\gamma_2^n\|_1 \leq & \left\{ \left(\|\alpha^n\| + \|\beta^n\| + \frac{3}{2} \right) \|u_{N_t}^n\| + \left(\frac{1}{2} + \|\beta^n\| \right) \|\eta_{N_t}^n\| \right\} + \frac{\Delta t}{2} \left\{ \|u_{N_{tt}}^n\| \right. \\ & \left. \|\alpha^n\| + \|\eta_{N_{tt}}^n\| \|\beta^n\| + \|u_{N_{tt}}^n\| \|\beta^n\| \right\} + \frac{\Delta t^2}{8} \left\{ \|\beta^n\| \|\alpha^n\| + \|\beta^n\|^2 \right\},\end{aligned}$$

$$\begin{aligned}\|\hat{\gamma}_1^n\|_1 + \|\hat{\gamma}_2^n\|_1 \leq & \left\{ \|\alpha^n\| \left(1 + \|u_N^n\|\right) + \frac{1}{2} \|\beta^n \left(1 + \|u_N^n\| + \|\eta_N^n\|\right) + \|u_{N_t}^n\| \|u_{N_{tt}}^n\| \right. \\ & \left. + \|\alpha_t^n\| + \frac{|a|}{2} \|\beta^n\|_2 + |c| \|\alpha^n\|_2 \right\} + \frac{\Delta t}{2} \left\{ \|\hat{\gamma}_1^n\|_1 + \|\hat{\gamma}_2^n\|_1 \right\},\end{aligned}$$

$$\begin{aligned}\|\gamma_3^n\|_1 + \|\gamma_4^n\|_1 \leq & \left\{ \left(1 + \|u_N^{n,4}\|\right) \|\hat{\gamma}_1^n\| + \left(1 + \|\eta_n^{n,4}\| + \|u_N^{n,4}\|\right) \|\hat{\gamma}_2^n\| + |a| \|\hat{\gamma}_1^n\|_2 + \right. \\ & |c| \|\hat{\gamma}_2^n\|_2 + 2\|\alpha^n\| \|u_{N_t}^n\| + \|\beta^n\| \left(\|u_{N_t}^n\| + \|\eta_{N_t}^n\| \right) + 2\|u_{N_{ttt}}^n\| \left(\|\eta_{N_t}^n\| + \right. \\ & \left. \left. \|u_{N_t}^n\| \right) + \|u_{N_{tt}}^n\| \left(2\|\eta_{N_{tt}}^n\| + \|u_{N_{tt}}^n\| \right) + 2\|u_{N_t}^n\| \|\eta_{N_{ttt}}^n\| \right\} + \Delta t \left\{ \|u_{N_{tt}}^n\| \right. \\ & \left. \left(\|\alpha^n\| + \|\eta_{N_{ttt}}^n\| \right) + \|\eta_{N_{tt}}^n\| \|u_{N_{ttt}}^n\| \right\} + \frac{\Delta t}{2} \left\{ \|\beta^n\| \left(\|\eta_{N_{tt}}^n\| + \|\eta_{N_{ttt}}^n\| \right) \right\} + \\ & \frac{\Delta t^2}{2} \left\{ \|u_{N_{ttt}}^n\| \left(\|\alpha^n\| + \|\eta_{N_{ttt}}^n\| \right) \right\} + \frac{\Delta t^2}{16} \left(\|\beta^n\|^2 + \Delta t^2 \|\hat{\gamma}_2{}^n\|^2 + 2\Delta t^2 \right. \\ & \left. \|\hat{\gamma}_1{}^n\| \|\hat{\gamma}_2{}^n\| \right) + \frac{\Delta t^2}{4} \left\{ \|\beta^n\| \left(\|\alpha^n\| + \|\eta_{N_{ttt}}^n\| + \|u_{N_{ttt}}^n\| \right) + \frac{1}{2} \|u_{N_{ttt}}^n\|^2 \right\},\end{aligned}$$

which are all bounded by constants, namely C, as consequences of $(ii.1)$-part of Lemma 12, Lemma 13 and estimates above.

These limitations are valid for Δt sufficiently small, and the constant in all the cases is of type $C = C(a, c)$, where a, c are parameters of the Boussinesq family system (4.56).

Remark 7. *Note that to prove the last estimate, we have used that* $\|\eta_N^{n,4}\| + \|u_N^{n,4}\| \leq C$*, which can be easily proved from the formulas to* $\eta_N^{n,4}$ *and* $u_N^{n,4}$ *given previously along with Lemma 12.*

Therefore, using the intermediate estimates given above, we can conclude that $\|\Gamma_1^n\|_1 + \|\Gamma_2^n\|_1 \leq C$. Therefore, using (4.155), we obtain that $\|\delta_1^n\|_1 + \|\delta_2^n\|_1 \leq C\Delta t^5$.

Indeed, expanding in Taylor series the functions $\eta_N(t)$ and $u_N(t)$ around $t = t_n$ and using Taylor theorem (see BURDEN; FAIRES (2010)), we get that

$$\eta_N(t) = \sum_{j=0}^{4} \frac{\partial_t^j \eta_N(t_n)}{j!}(t-t_n)^j + C(t-t_n)^5, \quad u_N(t) = \sum_{j=0}^{4} \frac{\partial_t^j u_N(t_n)}{j!}(t-t_n)^j + C(t-t_n)^5.$$

Evaluating these formulas at $t = t_{n+1}$, substituting in (4.155) and taking the H^1-norm, lead us to conclude (4.154), which ends the first part of the proof.

In sequence, we perform the stability part of the proof. Consider $\varepsilon^{n,j} := \eta_N^{n,j} - H^{n,j}$ and $e^{n,j} := u_N^{n,j} - U^{n,j}$ the errors between the solution of (4.56) evaluated at $t = t_n$ and the fully discrete approximations (H^n, U^n) in $S_N \times S_N$ of $\eta(.,t_n)$ and $u(.,t_n)$, solutions of (4.55), for $0 \leq n \leq M-1$ and $j = 1,2,3,4$.

Note that for $j = 1$ we have that $\varepsilon^n := \varepsilon^{n,1} = \eta_N^n - H^n$ and $e^n := e^{n,1} = u_N^n - U^n$. For $j = 2,3,4$ we get

$$\begin{aligned}
\varepsilon^{n,j} &= \varepsilon^n + \Delta t \alpha_i f(\varepsilon^{n,j-1}, e^{n,j-1}), \\
&= \varepsilon^n + \Delta t \alpha_i \left[f(\eta_N^{n,j-1}, e^{n,j-1}) + \hat{f}(\varepsilon^{n,j-1} u_N^{n,j-1}) + \hat{f}(\varepsilon^{n,j-1} e^{n,j-1}) \right], \\
e^{n,j} &= e^n + \Delta t \alpha_i g(\varepsilon^{n,j-1}, e^{n,j-1}), \\
&= e^n + \Delta t \alpha_i \left[g(\varepsilon^{n,j-1}, e^{n,j-1}) - \hat{g}(e^{n,j-1} e^{n,j-1}) + \hat{g}(e^{n,j-1} u_N^{n,j-1}) \right].
\end{aligned} \tag{4.161}$$

By hypothesis, it holds that $\max_{t\in[0,T]} \left(\|\partial_t^i \eta(t)\|_1 + \|\partial_t^i u(t)\|_1 \right) \leq \hat{M}$ for $i = 0,1,...,5$. Suppose that $\hat{M}$ is sufficiently large so that, for $i = 0,1,...,5$, $\max_{t\in[0,T]} \left(\|\partial_t^i \eta_N(t)\|_1 + \|\partial_t^i u_N(t)\|_1 \right) \leq 2\hat{M}$.

Let $n^* < \hat{M}$ be the largest integer for which $\|H^n\|_1 + \|U^n\|_1 \leq 3\hat{M}$ for $0 \leq n \leq n^*$. Hence, for $0 \leq n \leq n^*$ we have that

$$\begin{aligned}
&\|\varepsilon^n\|_1 + \|e^n\|_1 \leq 5\hat{M}, \\
&\|\varepsilon^{n,j}\|_1 + \|e^{n,j}\|_1 \leq (\|\varepsilon^n\|_1 + \|e^n\|_1) + C(\hat{M}, |a|, |c|)\Delta t \left(\|\varepsilon^{n,j-1}\|_1 + \|e^{n,j-1}\|_1 \right),
\end{aligned}$$

for $j = 2, 3, 4$

Furthermore, using the definition of ε and e and the RK4 method previously described, we can write that

$$
\begin{aligned}
\varepsilon^{n+1} &= \eta_N^{n+1} - H^{n+1}, \\
&= \eta_N^n + \Delta t \sum_{j=1}^{4} \beta_j f(\eta_N^{n,j}, u_N^{n,j}) + \delta_1^n - H^n - \Delta t \sum_{j=1}^{4} \beta_j f(H^{n,j}, U^{n,j}), \\
&= \varepsilon^n + \Delta t \sum_{j=1}^{4} \beta_j \left[f(\eta_N^{n,j}, u_N^{n,j}) - f(H^{n,j}, U^{n,j}) \right] + \delta_1^n,
\end{aligned}
\tag{4.162}
$$

and similarly,

$$
\begin{aligned}
e^{n+1} &= u_N^{n+1} - U^{n+1}, \\
&= e^n + \Delta t \sum_{j=1}^{4} \beta_j \left[g(\eta_N^{n,j}, u_N^{n,j}) - g(H^{n,j}, U^{n,j}) \right] + \delta_2^n.
\end{aligned}
\tag{4.163}
$$

Then, defining $A^n := \|\varepsilon^n\|_1 + \|e^n\|_1$, $A^{n,j} := \|\varepsilon^{n,j}\|_1 + \|e^{n,j}\|_1$ and summing the H^1-norms of (4.162) and (4.163), we get that

$$
A^{n+1} \leq A^n + \Delta t \sum_{j=1}^{4} A^{n,j} + \left(\|\delta_1^n\|_1 + \|\delta_2^n\|_1 \right).
$$

Since for Δt sufficiently small hold that $\Delta t \sum_{j=1}^{4} A^{n,j} \leq C \Delta t A^n$ and $\left(\|\delta_1^n\|_1 + \|\delta_2^n\|_1 \right) \leq C \Delta t^5$, we conclude that

$$
A^{n+1} \leq (1 + C\Delta t)\, A^n + C\Delta t^5, \quad A^0 = 0.
$$

Now, observe that

$$
A^{n+1} \leq \left[\frac{1 - (1 + C\Delta t)^{n+1}}{-C\Delta t} \right] C\Delta t^5 = \left[(1 + C\Delta t)^{n+1} - 1 \right] C\Delta t^4, \ 0 \leq n \leq n^*,
$$

and since we considering Δt sufficiently small, we can guarantee that $A^{n+1} \leq C(n^*)\Delta t^4$.

Finally, observing that we can write from the definitions of ε and e that $H^{n+1} = \eta_N^{n+1} - \varepsilon^{n+1}$ and $U^{n+1} = u_N^{n+1} - e^{n+1}$, we get that

$$\begin{aligned} \|H^{n+1}\|_1 + \|U^{n+1}\|_1 &\leq \| \left(\eta_N^{n+1}\|_1 + \|u_N^{n+1}\|_1\right) + (\varepsilon^{n+1}\|_1 + \|e^{n+1}\|_1), \\ &\leq 2\hat{M} + A^{n+1}, \\ &\leq 2\hat{M} + C(n^*)\Delta t^4 \leq 3\hat{M}, \end{aligned}$$

independently of n^* since Δt is small, contradicting the maximal property of n^*. Therefore, $\|H^n\|_1 + \|U^n\|_1 \leq C$ for all $0 \leq n \leq M$ and so,

$$\max_{0\leq n\leq M} (\|\varepsilon^n\|_1 + \|e^n\|_1) = \max_{0\leq n\leq M} (\|\eta_N^n - H^n\|_1 + \|u_N^n - U^n\|_1) \leq C\Delta t^4.$$

Using this result along (4.65) lead us to conclude that, for the weakly dispersive Boussinesq system case I, it holds the estimate (4.152). $\square$

A similar idea can be applied to the others cases, with small modifications. For example, to the case where $\mathcal{H}$ is of order 0, say $a < 0, b > 0, c = 0, d = 0$, we obtain the ODE system given by

$$\begin{aligned} \eta_{N_t} &= f(\eta_N, u_N), \\ u_{N_t} &= \overline{g}(\eta_N, u_N), \end{aligned} \tag{4.164}$$

for all $t \in [0, T]$. The functions $\hat{f}, \hat{g} : L^2 \to S_N$ are given respectively by $a_B(\hat{f}(v), \chi) = (v, \chi')$ and $a_D(\hat{g}(w), \chi) = (w, \chi')$ for all χ in S_N, with $f : H^1 \times H^2 \to S_N$ and $\overline{g} : H^1 \times H^1 \to S_N$ given respectively by $f(v, w) = \hat{f}(w) + \hat{f}(vw) + a\hat{f}(w'')$ and $\overline{g}(v, w) = \hat{g}(v) + \frac{1}{2}\hat{g}(w^2)$.

Observe that, since $\overline{g}$ is equal to g in the weakly case unless the term $c\hat{g}(v'')$, we can apply the same intermediate steps as before to the function $\overline{g}$, disregarding the appropriate term. Moreover, Lemma 13 still can be applied, since $\|\hat{g}(v)\| \leq \|\hat{g}(v)\|_1 \leq C\|v\|,\ v \in L^2$.

Then, we conclude that for this case we have the following estimate

$$\max_{0\leq n\leq M} \left(\|\eta(t^n) - H^n\|_1 + \|u(t_n) - U^n\|\right) \leq C\left(\Delta t^4 + N^{1-s}\right), \tag{4.165}$$

for $s \geq 3$.

Remark 8. *There is another possibility in this case, which is $a = 0, b = 0, c < 0, d > 0$. In this case, we consider the same function g defined before, but we replace the function f by the function $\overline{f} : H^1 \times H^1 \to S_N$ given by $\overline{f}(v, w) = \hat{f}(w) + \hat{f}(vw)$. Then, we proceed in the same way that described before, and we get that*

$$\mathit{max}_{0\leq n\leq M} \left(\|\eta(t^n) - H^n\| + \|u(t_n) - U^n\|_1\right) \leq C\left(\Delta t^4 + N^{1-s}\right). \tag{4.166}$$

In the case where $\mathcal{H}$ is of order 1, that is $a = 0, b = 0, c = 0, d > 0$, the idea is similar to the idea for the case when $\mathcal{H}$ is of order 0, but we consider $\overline{f}$ and $\overline{g}$ in the same time. After all the steps described, we obtain for this case an estimate similar to (4.166).

When $\mathcal{H}$ is of order 2, that is $a < 0, b = 0, c = 0, d > 0$, we apply the same ideas using f and $\overline{g}$. For this case, we also obtain an estimate similar to (4.166).

5 NUMERICAL EXPERIMENTS

In this chapter we present the results of several numerical experiments. First of all, we do some runs in order to validate our code implemented in MATLAB, and confirm the stability prediction obtained in section 4.2. These experiments are focused on the cases where the stability condition is given by $\Delta t \leq C$ or $\Delta t \leq CN^{-1}$, because the corresponding fully discretizations are not stiff and we can achieve a high accuracy at a low computational cost.

Second of all, we present numerical experiments showing two-way propagation of waves, specifically the resolution into, and the interaction of, solitary waves. The existence of such waves as solutions of nonlinear Boussinesq systems (3.31) should be expected at least in the range where this system approximates the Euler equations.

5.1 Validation of the Code

In order to validate our code, we computed the cnoidal waves solutions of the Bona-Smith system. A similar type of simulation was done in ANTONOPOULOS; DOUGALIS; MITSOTAKIS (2010b) in order to test the accuracy of the fully discrete Galerkin scheme developed there.

The existence of such solutions for the Boussinesq system (3.31) has been studied in CHEN; CHEN; NGUYEN (2007). The general family of cnoidal wave solutions of (3.31) is $\eta(\xi) = \eta_0 cn^2[\lambda\xi; m]$ and $u(\xi) = B\eta(\xi)$, where $\xi = x - c_s t$,

$$\begin{aligned}
&\alpha = \tfrac{8}{3\theta^2-1} - 6, \quad \beta = \tfrac{4}{3\theta^2-1} - 2, \quad \gamma = 2\theta^2 - 4/3,\\
&B^2 = \tfrac{2(b-c-2d)}{b-a-2d}, \quad c_s = \tfrac{2-B^2}{B},\\
&\eta_0 = (3 - 3B^2 + \sqrt{9 + 6(2A-3)B^2 + 9B^4})/2B^2,\\
&\kappa = \sqrt{9 + 6(2A-3)B^2 + 9B^4}/B^2, \quad \lambda = (\beta\kappa/6\gamma)^{1/2},\\
&m = (\eta_0/\kappa)^{1/2}.
\end{aligned}$$

Applying a procedure similar to the one used in CHEN (1998), Antonopoulos et al. derived in ANTONOPOULOS; DOUGALIS; MITSOTAKIS (2010b), exact solitary waves solutions of Bona-Smith system, where the parameters a, b, c, d are

$$a = 0, \quad c = (2 - 3\theta^2)/3, \quad b = d = (3\theta^2 - 1)/6, \quad 2/3 \leq \theta^2 \leq 1.$$

They also carried out its computation through the Galerkin-finite element method with periodic splines and the RK4 method for discretizing in space and time, respectively (see Fig. 2 of ANTONOPOULOS; DOUGALIS; MITSOTAKIS (2010b)).

In Figure 5.1 we show the evolution of the η component of the numerical solution for the nonlinear Bona-Smith system with $\theta^2 = 9/11$ using our fully discrete numerical method, which is composed by the Fourier collocation and fourth order RK methods. The Fourier collocation method was implemented in MATLAB using the Fast Fourier Transform (FFT) routine. We consider the same input used in ANTONOPOULOS; DOUGALIS; MITSOTAKIS (2010b) to run our simulation: spatial mesh with 240 intervals and a timestep $\Delta t = 10^{-2}$, $L = 1.82390$ and $A = 0.5$.

In figure 5.2 we show both the exact and the approximate solution generated by our code. The approximation error in L^2-norm at $t = 100$ is 9.3755×10^{-5}.

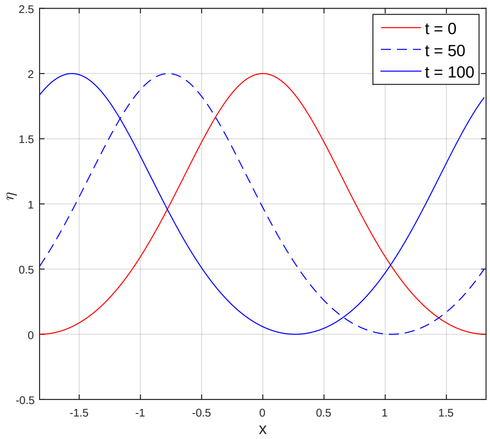

Figure 5.1: Evolution of an η-cnoidal wave using $\Delta t = 10^{-2}$ and $N = 240$.

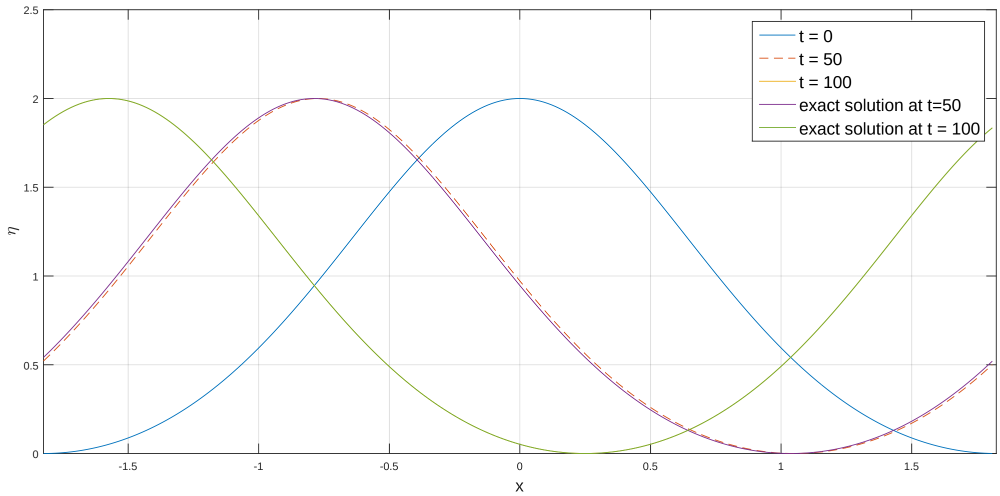

Figure 5.2: Exact vs approximate η-cnoidal wave.

We also computed the traveling waves solutions of the Bona-Smith system. A rigorous expression for this type of solution for this system and for several Boussinesq systems was established by Chen in CHEN (1998). These waves are not solitary waves, which for the Bona-Smith system were reported by Toland in TOLAND (1981), but they can be used to test the accuracy of numerical computations. A similar type of simulation was done in PELONI; DOUGALIS (2001) in order to test the accuracy of the fully discrete Fourier-Galerkin scheme developed there.

In order to show the accuracy of our numerical method, we used the traveling wave solution for Bona-Smith given by

$$\eta(\xi) = A \sinh^2\left(\frac{\sqrt{A}}{C_\tau}\xi\right), \quad u(\xi) = \frac{2}{C_\tau}\left(\eta(\xi)+1\right), \tag{5.1}$$

where the velocity of the wave, C_τ, is such that $|C_\tau| \geq 2$, $\xi = x - C_\tau t$ and $A = 3(C_\tau^2 - 4)/4$.

The Table 5.1 show the computed temporal error between the exacts η and u traveling wave solutions and the respective approximated solutions obtained by our code for the nonlinear Bona-Smith system for successively smaller values of Δt with fixed $N = 2^{10}$ and $A = 2.5$ in (5.1) at the time $t = 10$. The period L was considered large enough so that the initial wave could be considered practically periodic.

	η		u	
Δt	H^1 error	Conv. Rate	L^2 error	Conv. Rate
5e-2	6.3305e-5	-	1.0675e-5	-
2.5e-2	3.5375e-6	4.17	5.9702e-7	4.16
1.25e-2	2.0755e-7	4.09	3.5063e-8	4.08
6.25e-3	1.2545e-8	4.04	2.1209e-9	4.04

Table 5.1: Bona-Smith system errors and temporal convergence rates.

We also show the computed errors between the exacts η and u traveling wave solutions and the respective approximated solutions obtained by our code for the nonlinear KdV-KdV system, i.e., system (3.31) with $a = c = 1/6$ and $b = d = 0$.

In this case, we considered as initial conditions

$$\begin{aligned} \eta(\xi) &= \frac{1}{10}\sinh^2\left(\frac{\xi}{\sqrt{10}}\right) - \frac{16}{30}, \\ u(\xi) &= \frac{1}{5\sqrt{2}}\sinh^2\left(\frac{\xi}{\sqrt{10}}\right), \end{aligned} \tag{5.2}$$

where the velocity of the wave, $C_\tau = \dfrac{16}{15\sqrt{2}}$, and $\xi = x - C_\tau t$.

	η		u	
Δt	L^2 error	Conv. Rate	L^2 error	Conv. Rate
2.5e-2	1.8339e-11	-	4.8930e-10	-
2e-2	7.5138e-12	3.99	2.0046e-10	3.99
1.25e-2	1.1554e-12	3.98	3.0786e-11	3.98
1e-2	4.9108e-12	3.83	1.3012e-12	3.85

Table 5.2: KdV-KdV system errors and temporal convergence rates.

Both tables 5.1 and 5.2 confirm the theoretical temporal order of accuracy expected for the RK4 method, as explained in section 4.3.2. Observe that the space convergence is not verified; the reason of this is that spectral methods are exponentially accurate and we choose in both examples analytic initial conditions.

The numerical rate of convergence (fourth and fifth columns in tables 5.1 and 5.2) is determined by considering two different computations for the same problem at the same time t, with errors E_i and E_{i+1} corresponding to time steps Δt_i and Δt_{i+1}, respectively. The rate corresponding to these data is calculated by

$$\text{Rate} = \frac{\log(E_{i+1}/E_i)}{\log(\Delta t_{i+1}/\Delta t_i)}, \quad i = 1, \ldots, M-1,$$

since N is considered large enough to ensure negligible spatial error.

It is also worth observe that the choice of norms for the error reflects the natural choices of pair of spaces in which the respective systems are linearly well-posed.

5.2 Numerical Stability of the Linear Problem

In this part, we show the numerical verification of the stability condition for the linear Boussinesq system with some particular choices of the parameters a, b, c, d, more specifically the ones with $\ell = 0, 1$, according to Table 4.1. All computations were performed with $L = 150$, $T = 70$ and the number of elements of the spatial mesh as a power of 2.

	$a = -7/30, b = 7/15,$ $c = -2/5, d = 1/2$		$a = b = c = d = 0$	
N	Δt	Stability constant C	Δt	Stability constant C
2^9	0.8060	1.37	0.5313	0.90
2^{11}	0.2102	1.43	0.1328	0.90
2^{13}	0.0523	1.42	0.0330	0.90
2^{15}	0.0130	1.41	0.0082	0.89

Table 5.3: Numerical stability constants.

	$a = -7/30, b = 7/15,$ $c = 0, d = 0$		$a = 0, b = 0,$ $c = -2/5, d = 1/2$	
N	Δt	Stability constant C	Δt	Stability constant C
2^9	0.7273	1.24	0.5892	1.00
2^{11}	0.1879	1.28	0.1485	1.01
2^{13}	0.0468	1.27	0.0369	1.00
2^{15}	0.0116	1.26	0.0092	1.00

Table 5.4: Numerical stability constants.

In Tables 5.3 and 5.4, the second and fourth columns indicate the last Δt such the numerical solution was stable. The third and fifth columns indicates the numerical stability constant C, which was calculated using that since $\Delta x = 2L/N$, then $C \simeq \frac{\Delta t}{\Delta x}$ for these four regions of parameters, according with Table 4.1. We use as initial conditions for these simulations two Gaussian pulses, namely $\eta_0 = e^{-5x^2}$ and $u_0 = -e^{-5x^2}$.

In Table 5.5 we show the same verification for two regions of parameters such that the stability condition is of type $\Delta t \leq C$, according to Table 4.1. The input values of the code in these cases are the same that the ones used for obtain the Tables 5.3 and 5.4. In this case, the numerical stability constant C does not depend on the variation of Δx, as can be observed.

	$a=-7/30, b=7/15,$ $c=0, d=1/2$		$a=0, b=7/15,$ $c=0, d=0$	
N	Δt	Stability constant C	Δt	Stability constant C
2^9	2.8	2.8	2.0	2.0
2^{11}	2.8	2.8	1.9	1.9
2^{13}	2.8	2.8	1.9	1.9
2^{15}	2.8	2.8	1.9	1.9

Table 5.5: Numerical stability constants.

The criteria to determine the stability of the numerical approximation for η is the following: we evaluate the L^2-norm of the initial data, $\|\eta_0\|_{L^2(-L,L)}$, and compare with the same norm of the approximation at $t = 70$ obtained by the numerical scheme, $\|\eta\|_{L^2(-L,L)}$. If

$$p = \frac{\|\eta\|_{L^2(-L,L)}}{\|\eta_0\|_{L^2(-L,L)}} < C_1,$$

where C_1 is a constant slightly greater than 1, we consider the numerical solution stable.

5.3 Numerical Stability of the Nonlinear Problem

In this section, we perform some stability tests for the nonlinear Boussinesq system (3.31) assuming the system is weakly nonlinear. More specifically, we consider the system

$$\begin{aligned} \eta_t + u_x + \alpha(u\eta)_x + au_{xxx} - b\eta_{xxt} &= 0, \\ u_t + \eta_x + \alpha uu_x + c\eta_{xxx} - du_{xxt} &= 0, \end{aligned} \tag{5.3}$$

in which α is a small positive constant.

Figure 5.3 contains two η-waves, approximate solutions of the nonlinear system (5.3) for different choices of α. We run the code considering the parameters $a = -7/30$, $b = 7/15$, $c = -2/5$, $d = 1/2$ and $N = 2^{13}$. The last Δt such that the numerical solution was stable in this case is $\Delta t = 0.052$, as Table 5.3 suggests. We can observe that the numerical solutions considering $\alpha = 10^{-1}$ and $\alpha = 0$ are slightly different.

Observe that the later assumption leads to the linear system analyzed in section 5.2. This results suggests that the stability of the numerical solution of the weakly nonlinear system barely depends on α in this case.

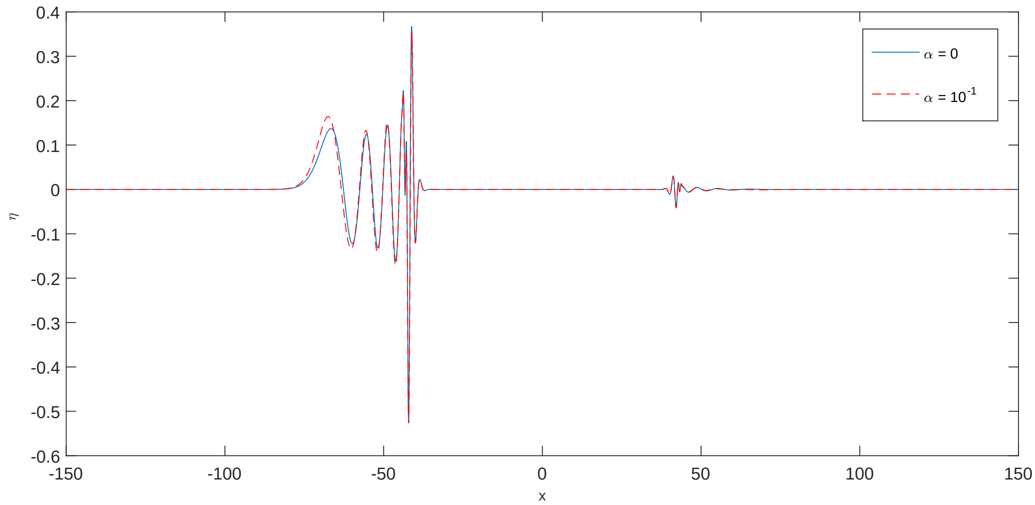

Figure 5.3: Comparison of the nonlinear η solution in $t = 70$ for two values of α.

Figure 5.4 also contains two η-waves, approximate solutions of the nonlinear system (5.3) for different choices of α. But now, our aim is observe how the non linearity affects the stability of the numerical solution for the regions of parameters represented in Table 5.5. First of all, we run the code considering the parameters

$a = b = d = 0$, $b = 7/15$ and $N = 2^{13}$. The last Δt such that the numerical solution was stable in this case is $\Delta t = 1.9$, as Table 5.5 suggests, and we can observe that the numerical solutions considering $\alpha = 10^{-3}$ and $\alpha = 0$ are practically the same.

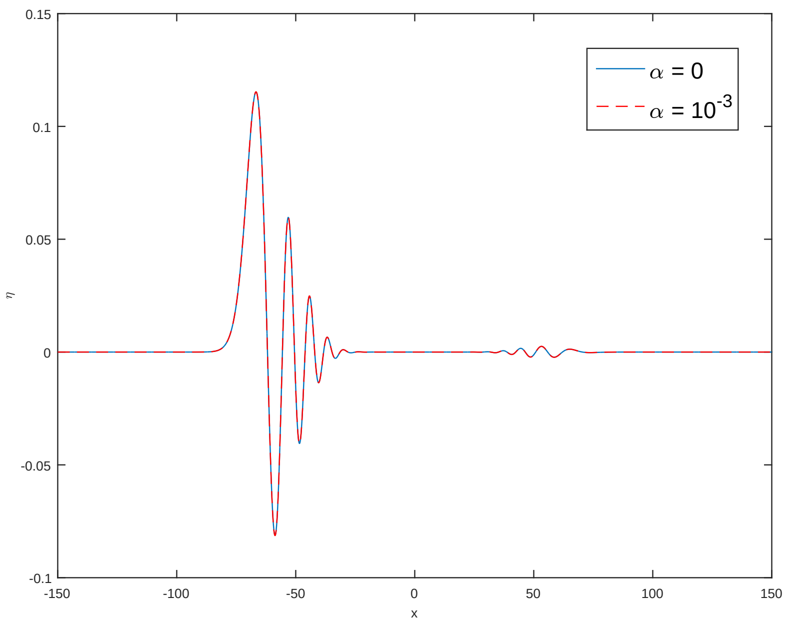

Figure 5.4: Comparison of the nonlinear η solution in $t = 70$ for two values of α.

On the other hand, when we run the code considering the same input data as before, i.e., the parameters $a = b = d = 0$, $b = 7/15$ and $N = 2^{13}$, but using $\alpha = 10^{-1}$ in (5.3), there is a big change in the stability of the numerical solution. This numerical fact can be observed in Table 5.6, by the variation of the parameter p introduced in section 5.2, which is responsible to determine if a solution is stable or not.

We observe in Table 5.6 that, if we consider $\Delta t = 1.7$, the last Δt producing a stable solution for this case with $N = 2^{13}$, as we could expect by the right column of Table 5.5, the value of p goes to infinity. We tested smaller values for Δt in this case, in fact until $\Delta t = 10^{-2}$, but the solution remained unstable, with the parameter p

N	Δt	p
2^9	2.0	0.3756
2^{11}	1.7	0.3856
2^{13}	1.7	NaN (Not a Number)

Table 5.6: Numerical stability constants.

assuming large values.

Another interesting observation is that, when we run the code considering the parameters $a = -7/30$, $b = 7/15$, $c = 0$ and $d = 1/2$, the stability of the numerical solution when $\alpha = 1$, which is equivalent to the nonlinear original system, does not change. In other words, the left column of the Table 5.5 is maintained for the nonlinear problem even with a slight change on their plots. This fact is illustrated in Figure 5.5.

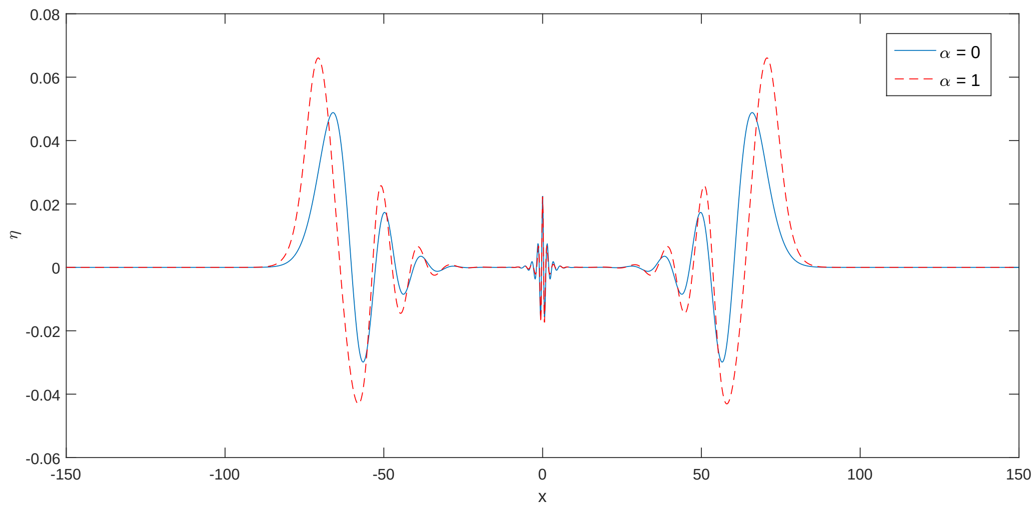

Figure 5.5: Comparison of the nonlinear and linear η solution at $t = 70$.

5.4 Solitary Waves

In this section, we present some numerical experiments showing two-way propagation of waves, more specifically, the resolution into, and interaction of, solitary waves.

5.4.1 Two-way Propagation and Resolution property

By resolution into solitary waves we mean the following property: an arbitrary profile for the elevation η of large enough L^2 norm, which means that the norm of the initial wave must be at least equal to the norm of the solitary wave of the same amplitude for the system considered, initially at rest, is resolved after some time, into one or more solitary waves, traveling without being subject to further alterations, plus a dispersive tail. This property has been observed in one-way propagation models, and it is also observed in our numerical experiments for the nonlinear Boussinesq systems.

We illustrate this property performing two simulations. We take as initial conditions a well localized wave form with zero initial velocity, i.e., $u_0 = 0$. Figure 5.6 shows the two-way propagation generated by the classical Boussinesq system, considering as initial condition the Gaussian pulse $\eta_0(x) = 0.5e^{-(x/3)^2}$, $N = 256$, $L = 50$ and $\Delta t = 0.1$. The result is plotted up to time $t = 20$.

We can observe in figure 5.6 that the initial wave divides itself in two waves with approximately half of the amplitude, traveling in opposite directions, plus two dispersive tails each one of them.

We observe a similar behavior when we evolve a Gaussian pulse under the

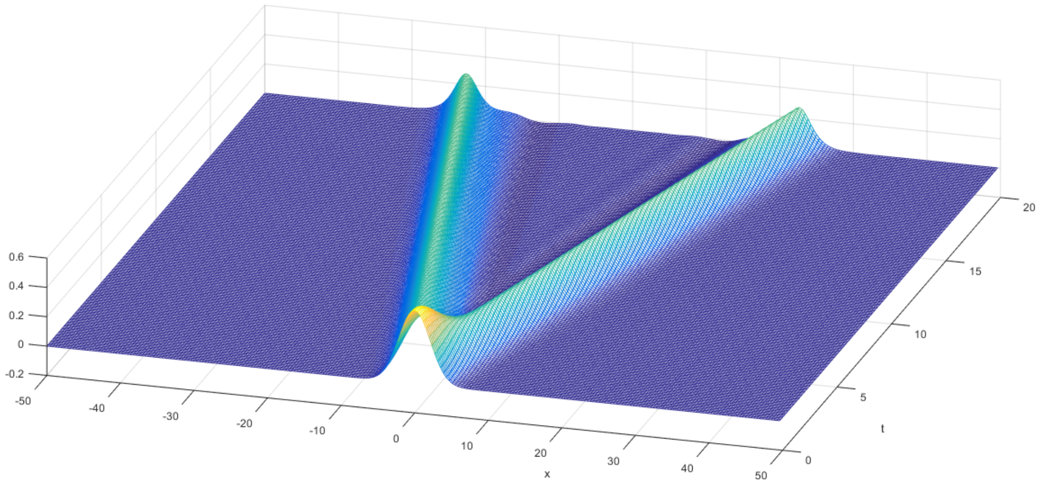

Figure 5.6: Two-way propagation of classical Boussinesq system.

action of the Bona-Smith system. Figure 5.7 shows the resulting propagating waves in this case. We considered as initial condition $\eta_0(x) = 1.5e^{-(x/2)^2}$, $N = 128$, $L = 150$ and $\Delta t = 0.1$.

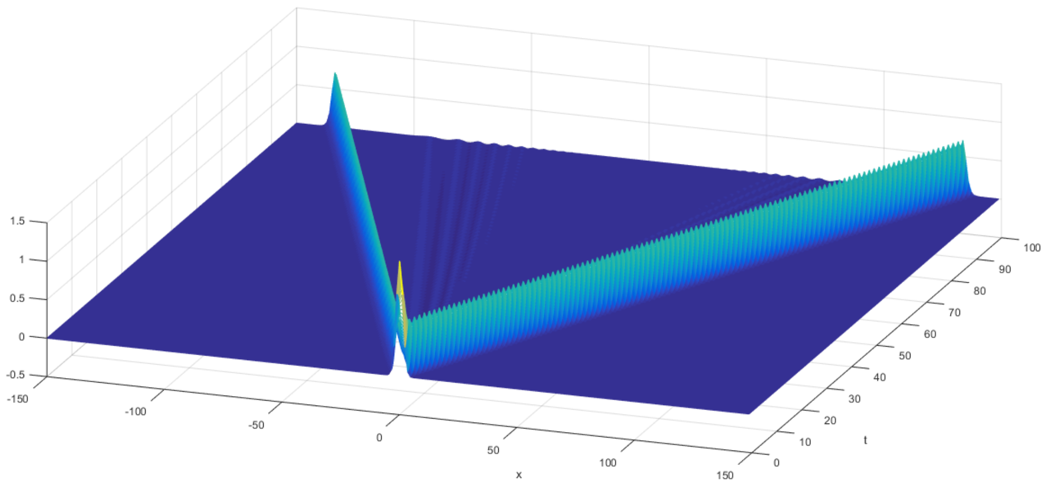

Figure 5.7: Two-way propagation of Bona-Smith system.

Considering more closely the resolution property, we observe that the larger the norm of the initial profile η, the more solitary waves are resolved. However, the relation between the initial norm and the number of solitary waves produced is not known for Boussinesq systems in general. This question is partially settled for some of the integrable one-way propagating models.

We give in the following, another example of resolution for the classical Boussinesq system. Figure 5.8 shows the resolution of the Gaussian pulse $\eta_0(x) = 2e^{-(x/5)^2}$, considering $u_0 = 0$, $N = 512$, $L = 150$ and $\Delta t = 0.025$.

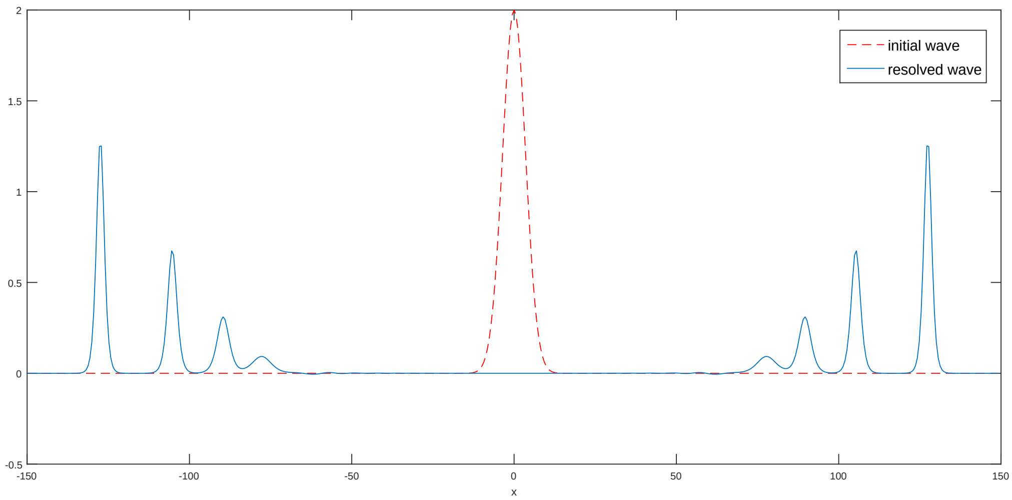

Figure 5.8: Classical Boussinesq sytem, resolution of a Gaussian at $t = 100$.

All the numerical experiments performed in this section suggest that solitary wave solutions of Boussinesq systems, in particular, the last considered systems, although not known analytically, can be produced numerically. This is done in BONA; CHEN (1998) for the particular BBM-BBM system. The proceeding developed there is the following: consider an initial profile resembling a solitary wave with null initial velocity; let this profile evolve under the action of the considered system; then, isolate numerically the leading pulse on one side by setting the remainder of the solution equal to zero; the resulting wave is thus used as the new initial wave form.

Repeating this procedure, the authors were able to produce a clean pulse which propagated with hardly any further change, which thus was taken as a good approximation to an exact solitary wave solution.

In our case, the Fourier collocation method allows us to repeat this procedure for other Boussinesq systems as well. In order to illustrate this process, we give an example of the wave obtained by this procedure for the classical Boussinesq system.

In figure 5.9, we show the generation of a solitary wave, numerically isolated after the resolution experiment, using the procedure described before iterated twice. We start with the Gaussian $\eta_0(x) = e^{-x^2}$ initially at rest, considering $N = 1024$, $L = 150$, $\Delta t = 0.05$ and $t = 80$. When two peaks, traveling to left and the right side, respectively, separated from the remaining part of the solution, we choose the left-traveling one and set the rest of the solution and of its velocity equal to zero. This numerically isolated peak was thus used as the new initial wave η_0, with u_0 equal the velocity of the left-traveling pulse that we isolated, with sign reversed, so that the wave propagates to the right.

Repeating this procedure one more time, the evolution that we obtained after the second iteration did not produce a visibly appreciable oscillatory tail. Therefore, we conclude that this profile is close to a solitary wave solution. Indeed, after evolving up to $t = 80$, it had not changed its shape, and the oscillation produced is roughly of the order of thickness of the line in the plot. The amplitude of the wave evolved in this figure is $A_0 = 0.2951$ at $t = 0$, and $A_1 = 0.2950$ at $t = 80$.

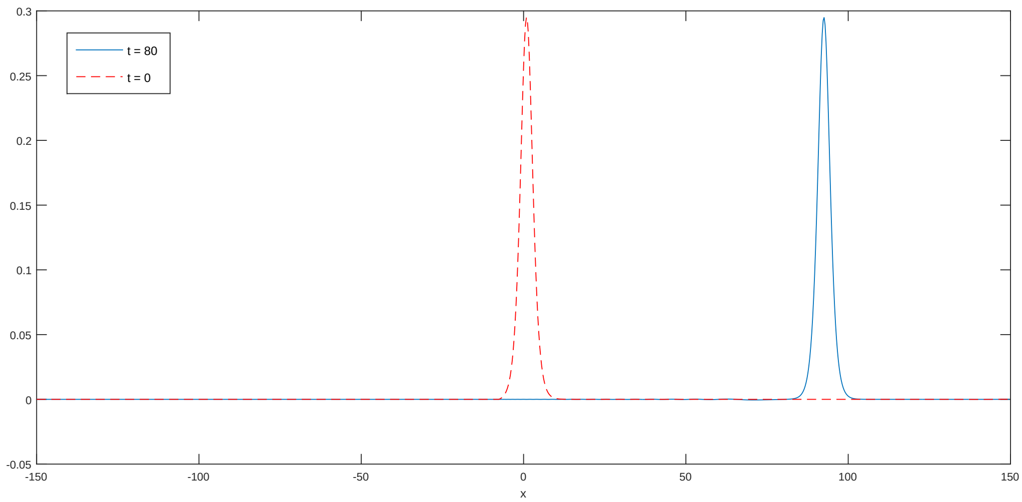

Figure 5.9: Evolution of a numerical solitary wave solution for classical Boussinesq system, from $t = 0$ to $t = 80$.

5.4.2 Interaction of Solitary Waves

In this section, we give an example of interaction of solitary waves. The solitary waves we use are the ones produced numerically as explained in section 5.4.1. The interaction of two solitary waves traveling in oposite directions has been studied in detail for the BBM-BBM system in BONA; CHEN (1998) as well.

We consider the interaction of two solitary waves of the classical Boussinesq system with same amplitude and traveling in opposite directions.

In figure 5.10, we show two classical Boussinesq solitary waves, numerically isolated using the same initial data used in section 5.4.1. The two numerical solitary waves are initially of amplitude $A_0 = 0.2950$ and centered at $x = \pm 92.5781$.

In figure 5.11, the waves are shown at $t = 0$, at the moment of interaction

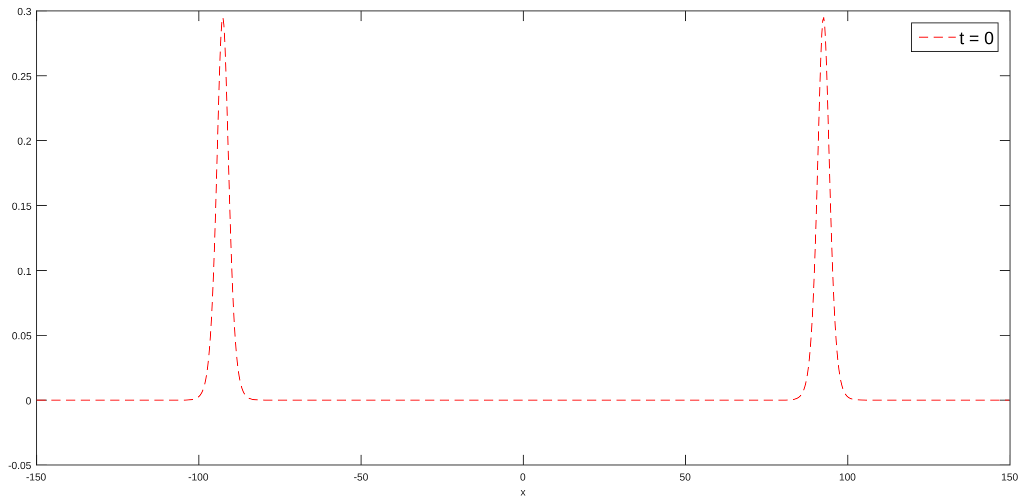

Figure 5.10: Two numerical solitary wave solutions for classical Boussinesq system at $t = 0$, with amplitude $A_0 = 0.2950$.

$t = 80$ and at $t = 140$ after they have interacted, and have recovered their original shape. Figure 5.12 shows the initial and the final profile after the interaction; in this figure, we can observe the dispersive tail, as it is expected for this type of systems. Finally, in Figure 5.13 we see a magnification of the same wave at $t = 100$.

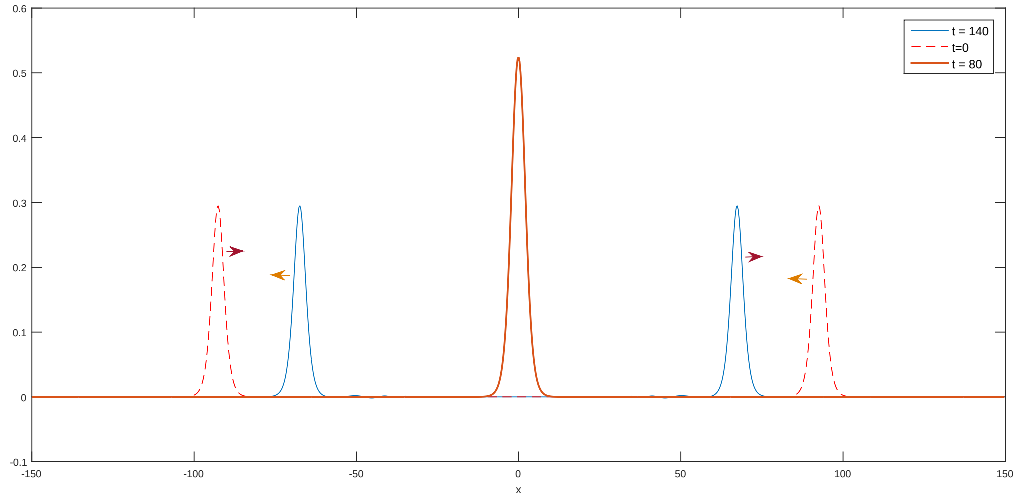

Figure 5.11: Solitary waves of Clasical Boussinesq system interating.

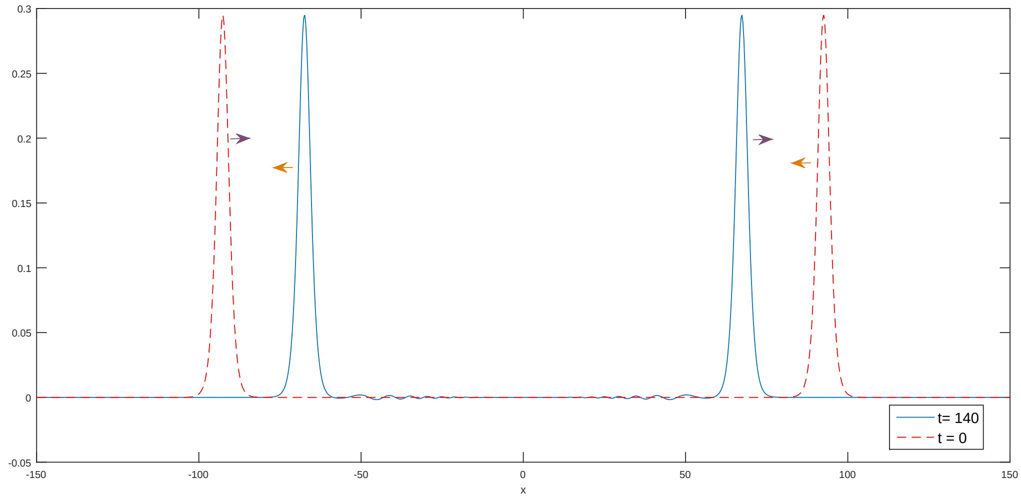

Figure 5.12: Solitary waves of Clasical Boussinesq system interating.

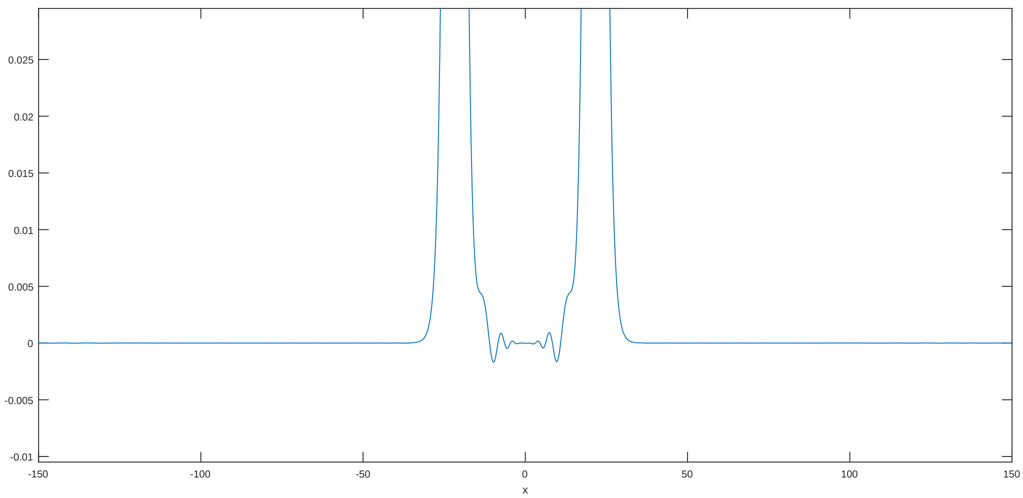

Figure 5.13: Magnification of the tail after the interaction at $t = 100$.

6 CONCLUSIONS

We have studied the numerical stability of fully discrete scheme for a linear Boussinesq systems. Section 4.2 provided the von Neumann analysis of the linear model obtained from system (3.31).

This analysis allowed us to identify which regions of the parameters a, b, c, d of the linear Boussinesq system were capable to generate numerical solutions more efficiently. Efficiency in the sense that even with a bigger time step Δt, the stability of the linear numerical solution was not lost.

In section 5.3 we have shown some nonlinear examples, concluding that the region of stability expected for the linear problem was preserved if the system is barely nonlinear, i.e., when the nonlinearity of the system is small and controllable. The classification of the type of stability condition given by Table 4.1 showed what regions of parameters give rise to problems with smaller computational cost for the numerical resolution. Moreover, the comments did in section 5.3 suggests that, even the stability analysis was performed for the linear problem, we can expect something similar under some conditions for the nonlinear problem.

The analysis and the examples provided in sections 5.2 and 5.3 suggest that the previously knowledge of what regions of parameters gives rise to problems with simpler numerical resolution can serve as guideline to study more complex problems numerically.

In section 5.1 we checked the temporal order of convergence expected by the application of the RK4 method. Moreover, we also performed the numerical study of solitary waves for these systems as an application of the numerical analysis. The sections 5.4.1 and 5.4.2 suggest that we are capable to generate numerically solitary waves for these systems and to study their interaction.

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